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Automorphism Modular Invariants of Current Algebras

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Abstract: We consider those two-dimensional rational conformal field theories (RCFTs) whose chiral algebras, when maximally extended, are isomorphic to the current algebra formed from some affine non-twisted Kac–Moody algebra at fixed level. In this case the partition function is specified by an automorphism of the fusion ring and corresponding symmetry of the Kac–Peterson modular matrices. We classify all such partition functions when the underlying finite-dimensional Lie algebra is simple. This gives all possible spectra for this class of RCFTs. While accomplishing this, we also find the primary fields with second smallest quantum dimension.

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1. Introduction

In two–dimensional conformal field theory, scale invariance means boundary conditions have an impact on the local physics, even far from a boundary [6]. For example, a conformal field theory must be consistent on the interior of a parallelogram with periodic boundary conditions imposed, i.e. on a torus. In particular, the corresponding partition function should not be sensitive to changes of the modular parameter that keep a torus within the same conformal class. The partition function must be modular invariant.

The local symmetry of the conformal field theory also constrains the partition function. The chiral algebra of currents determines the conformal blocks [2] of the torus partition function. That is, the partition function must be a sesquilinear combination of characters of the chiral algebra. The two constraints together often determine completely the field content of a given conformal field theory. This analysis of conformal field theories is known as the modular bootstrap.

We apply the modular bootstrap program to conformal field theories whose (maximal) chiral algebras are isomorphic to the current algebra of nontwisted affine Kac–Moody algebras at fixed levels. We call such algebras *conformal current algebras*, and the corresponding theories *unextended current models*. Their partition functions are described by a permutation matrix that also gives the action of a fusion rule automorphism [27]. For this reason, candidates for such partition functions are known as automorphism (modular) invariants. We will limit our attention here to the case where the underlying finite-dimensional Lie algebra is simple.

We actually solve the slightly more general problem of finding, for each simple Lie algebra X_ℓ and level k, the set of all permutations σ of the alcôve $P_+(X_{\ell,k})$ of highest weights (see (3.1) below), which are symmetries of the corresponding Kac-Peterson modular matrices, i.e. which obey equations (3.8a),(3.8b) below. This is what we mean by *automorphism invariants*. Their classification should be of mathematical value independent of RCFT. The question of which of these are actually realized as the partition function of a RCFT is not addressed here.

There are in the literature at least two different meanings of the phrase *Wess-Zumino-Witten (WZW) models*. The more general one is any RCFT whose maximal chiral algebras contain a conformal current algebra such that any character of the former can be written as a finite sum of characters of the latter. The partition function for such a RCFT will then be a finite sesquilinear combination of affine algebra characters. We suggest the term *current models* for these; when the chiral algebra equals the current algebra, we will call them *unextended* current models. A more restrictive definition are those RCFTs corresponding to a string moving on a compact Lie group [18] — we retain the term *WZW model* for these. The WZW partition function has been computed for each simple, compact, connected Lie group [13]. In this paper we find all automorphism invariants; our list will include all possible partition functions for the unextended current models. Many of these automorphism invariants cannot be found in [13], and some still lack such an explicit interpretation.

The list of partition functions of unextended current models is presented below, and proved complete. This result is a major step towards the more ambitious classification of *all* modular invariants of current models, including those described by a chiral algebra that extends the conformal current algebra. The list we find is also the useful one from the point of view of symmetry. It is often easier to identify the symmetry of a physical theory, before identifying the details of the dynamics. In that sense, a list of possible partition functions with a given maximal chiral algebra is the most relevant. Our catalogue gives the complete list for each (simple) conformal current algebra.

This work follows [17], where the automorphism invariants for algebra $A_{m_1} \oplus \cdots \oplus A_{m_s}$ were treated. Our restriction here to simple Lie algebras is convenient, but as [17] shows, the generalization to semi-simple Lie algebras should be possible. It is hoped that classification results like ours will teach us something about more general classes of conformal field theories, perhaps all rational ones. The greatest impetus to this program was given by the curious A–D–E classification of A_1 modular invariants [5]. Extension of this work proved difficult: the A_2 invariants were only recently classified in [16] (special cases of the A_2 classification were also obtained in [29]). Because we treat all simple Lie algebras here, our work may reveal new features of these modular invariants families that are universal (previously, only the level one theories had been

classified for all simple Lie algebras [21,15]).

Our results are stated in section 2, along with a brief outline of the classification proof. Sections 3 through 7 are devoted to the proof. A short conclusion is given in section 8.

2. Statement of the Results

The paper will be devoted to the proof of the following statement, already proved for the A_ℓ series in [17].

Theorem. *The complete list of automorphism invariants σ for the current algebra $X_{\ell,k}$, where X_ℓ is a simple Lie algebra and $k \in \mathbb{N}$, is given in Tables 1 and 2.*

Explicit definitions of all automorphism invariants listed in Tables 1 and 2 are given by the relevant subsections of sections 5,6 and 7. The total number of automorphism invariants for fixed $X_{\ell,k}$ is given in the third column. These form a group under composition $\sigma \circ \sigma'$; this group is given in the final column of the tables.

From Tables 1 and 2, we see that the automorphism invariants can all be described solely in terms of the symmetries of the extended Dynkin diagram (conjugations and simple currents — see section 3), with the exceptions of $B_{\ell,2}$, $D_{\ell,2}$, $E_{8,4}$, $F_{4,3}$ and $G_{2,4}$, where exceptional automorphism invariants appear. Except for the $E_{8,4}$ one, these exceptionals stem from Galois transformations, with a subtle touch of simple currents — this will be discussed in more detail in section 3.

Although scattered in the literature, all simple current automorphism invariants have been known for some time [3,1,13,30]. The exceptional automorphism invariant of $E_{8,4}$ was first given in [10], while those of $F_{4,3}$ and $G_{2,4}$ were found in [35]. Finally all the exceptional automorphism invariants of $B_{\ell,2}$ and $D_{\ell,2}$ have been recently unveiled in [12], though no explicit description was given. Let us stress that all but one (namely the $E_{8,4}$ exceptional) of the automorphism invariants of Tables 1 and 2 can be fully accounted for in terms of simple currents, conjugations and Galois transformations. This is somewhat fortunate as they are the main systematic procedures to construct automorphism invariants.

Our proof of this theorem relies on three basic steps. See the following section for terminology.

We first examine the quantum dimensions $\mathcal{D}(\lambda) := S_{0,\lambda}/S_{0,0}$ for all weights in the alcôve. Let $[\lambda]$ denote the set of all transforms — the orbit — of λ by the symmetries of the extended Dynkin diagram; $\mathcal{D}(\lambda)$ is constant along $[\lambda]$. It is well known that, as a function of λ, $\mathcal{D}(\lambda)$ takes its minimal value 1 if and only if $\lambda \in [0]$ (for $E_{8,2}$ there is an additional such λ, hence an additional simple current, but it plays no role here and will be ignored). Thus $\mathcal{Q}_1 = [0]$ is the set of weights at which $\mathcal{D}(\lambda)$ is minimum. The first step of our proof is to look for the set $\mathcal{Q}_2$ of all weights at which $\mathcal{D}(\lambda)$ takes its second smallest value. In the generic case, we find that $\mathcal{Q}_2 = [\omega^f]$ for the fundamental weight ω^f of X_ℓ which has the smallest Weyl–dimension, in agreement with the large k limit of the quantum dimensions. If however the level k is sufficiently small, this simple statement may break down, as Table 3 shows — a prime example of that is given by the orthogonal algebras at level 2. In these cases however, the spurious possibilities can be handled by the norm condition (3.8a) and/or by looking at the sets $\mathcal{Q}_i$ for $i \geq 3$, except for $B_{\ell,2}$ and $D_{\ell,2}$, which require a special analysis. We refer the reader to the text for the details of these cases. When $\mathcal{Q}_2 = [\omega^f]$, we obtain our first conclusion that any automorphism σ must map the orbit $[\omega^f]$ onto itself.

If $\mathcal{Q}_2 = [\omega^f]$, we obtain from the first step that the action of σ on ω^f is of the form $\sigma(\omega^f) = C'J(\omega^f)$ for some conjugation C' and simple current J. A conjugation C' always defines an automorphism invariant, so that replacing σ by $C' \circ \sigma$ permits us to assume $\sigma(\omega^f) = J(\omega^f)$. Requiring that σ commute with the modular matrix T — the norm of the weights must be preserved — puts various restrictions on J, depending on the level k and the algebra we consider. Two situations are then possible. The first is that, for a given simple current J satisfying the norm condition, there does exist a simple current automorphism invariant σ' such that $\sigma'(\omega^f) = J(\omega^f)$. In this case, the action of J on ω^f lifts to an acceptable solution σ' on the whole of the alcôve. This means one may replace σ by $\sigma'^{-1} \circ \sigma$, and assume that σ fixes ω^f. The second situation

$X_{\ell,k}$	conditions	# autom.	names	group
$A_{\ell,k}$		2^{c+p+t}	$\{C^a\sigma_m\}$	$\mathbb{D}_1^{c+p+t}$
$B_{\ell,1}$		1	$\{\sigma_1\}$	
$B_{\ell,2}$		2^{p-1}	$\{\sigma_m\}$	$\mathbb{D}_1^{p-1}$
$B_{\ell,k}$	$k \geq 3,\ k$ odd	2	$\{\sigma_1, \sigma_J\}$	$\mathbb{D}_1$
$B_{\ell,k}$	$k \geq 4,\ k$ even	1	$\{\sigma_1\}$	
$C_{2,1}$		1	$\{\sigma_1\}$	
$C_{\ell,k}$	$k\ell \equiv_4 2,\ (\ell,k) \neq (2,1)$	2	$\{\sigma_1, \sigma_J\}$	$\mathbb{D}_1$
$C_{\ell,k}$	$k\ell \not\equiv_4 2$	1	$\{\sigma_1\}$	
$D_{\ell,1}$	$\ell \equiv_8 4$	6	$\langle\sigma_s, \sigma_c\rangle$	$\mathbb{D}_3$
$D_{\ell,1}$	$\ell \not\equiv_8 4$	2	$\{\sigma_1, C_1\}$	$\mathbb{D}_1$
$D_{4,2}$		6	$\{C_j\}$	$\mathbb{D}_3$
$D_{4,k}$	$k > 2,$ even	12	$\{C_j\sigma_{vsc}^a\}$	$\mathbb{D}_6$
$D_{4,k}$	$k > 1,$ odd	36	$\{C_j\langle\sigma_s, \sigma_c\rangle\}$	$\mathbb{D}_3^2$
$D_{\ell,2}$	$\ell > 4$	2^p	$\{C_1^a\sigma_m\}$	$\mathbb{D}_1^p$
$D_{\ell>4,k>2}$	k, ℓ even, and $k\ell \equiv_8 0$	4	$\{C_1^a\sigma_{vsc}^b\}$	$\mathbb{D}_2$
$D_{\ell,k}$	ℓ odd, $k \equiv_4 0$	2	$\{\sigma_1, C_1\}$	$\mathbb{D}_1$
$D_{\ell,k>2}$	ℓ odd, $k \equiv_4 2$	4	$\{C_1^a\sigma_s^b\}$	$\mathbb{D}_2$
$D_{\ell,k}$	$k > 1$ odd, $\ell \not\equiv_8 4$	4	$\{C_1^a\sigma_v^b\}$	$\mathbb{D}_2$
$D_{\ell,k>2}$	$k \equiv_4 \ell \equiv_4 2$	8	$\{C_1^a\langle\sigma_s, \sigma_c\rangle\}$	$\mathbb{D}_4$
$D_{\ell>4,k}$	$k > 1$ odd, $\ell \equiv_8 4$	12	$\{C_1^a\langle\sigma_s, \sigma_c\rangle\}$	$\mathbb{D}_6$

Table 1. Complete list of automorphism invariants for classical simple Lie algebras. The variables c, p, t for $A_{\ell,k}$ are defined in the text (section 5). For $B_{\ell,2}$, respectively $D_{\ell,2}$, p is the number of distinct prime divisors of $2\ell+1$, respectively ℓ. The exponents a, b range over $\{0,1\}$. We denote a congruence modulo m by $\equiv_m$. In the last column giving the structure of the automorphism group, we have denoted by $\mathbb{D}_m$ the dihedral group of order $2m$.

is when J does not lift to a simple current automorphism invariant.

The third and final step aims at filling the gaps left by the second step. On the one hand, we classify the automorphisms which leave ω^f fixed. When combined with the automorphisms which do not leave ω^f fixed — these were collected at Step 2 —, they yield the full set of automorphisms. On the other hand, we show that the possibilities $\sigma(\omega^f) = J(\omega^f)$ in the second situation in Step 2 cannot be extended globally to any automorphism invariant. The main tool to obtain these two results is the explicit computation of fusion products. Indeed a happy feature of ω^f is that it is sufficiently small and simple to allow the calculation of its fusion product with any other field, and this is what is basically needed though in some cases the fusion with other small representations is also required.

The first and crucial step of our proof is detailed in section 4, while the other two are worked out in section 5 for the classical algebras, and in section 7 for the exceptional ones. A section 6 is inserted that contains the relevant analysis for $B_{\ell,2}$ and $D_{\ell,2}$. For completeness, we include the results (but not the proofs) for the A_ℓ series [17].

$X_{\ell,k}$	conditions	# autom.	names	group
$E_{6,k}$	$k<3$ or $k\equiv_3 0$	2	$\{C^a\}$	$\mathbb{D}_1$
$E_{6,k>2}$	$k\equiv_3 \pm 1$	4	$\{C^a\sigma_J^b\}$	$\mathbb{D}_2$
$E_{7,k}$	$k=2$ or $k\not\equiv_4 2$	1	$\{\sigma_1\}$	
$E_{7,k}$	$k>2$ and $k\equiv_4 2$	2	$\{\sigma_1,\sigma_J\}$	$\mathbb{D}_1$
$E_{8,4}$		2	$\{\sigma_1,\sigma_{e8}\}$	$\mathbb{D}_1$
$E_{8,k}$	$k\neq 4$	1	$\{\sigma_1\}$	
$F_{4,3}$		2	$\{\sigma_1,\sigma_{f4}\}$	$\mathbb{D}_1$
$F_{4,k}$	$k\neq 3$	1	$\{\sigma_1\}$	
$G_{2,4}$		2	$\{\sigma_1,\sigma_{g2}\}$	$\mathbb{D}_1$
$G_{2,k}$	$k\neq 4$	1	$\{\sigma_1\}$	

Table 2. Complete list of automorphism invariants for exceptional simple Lie algebras. The exponents a,b range over $\{0,1\}$. A congruence modulo m is denoted by $\equiv_m$. The notation $\mathbb{D}_m$ stands for the dihedral group of order $2m$.

3. Notations and Preliminaries

Let X_ℓ be a finite–dimensional simple Lie algebra. The weights are denoted in the Dynkin basis by $\lambda = (\lambda_1,\lambda_2,\ldots,\lambda_\ell) := \sum_i \lambda_i\omega^i$ with all λ_i integers, where ω^i is the i–th fundamental weight. (Our convention for the numbering of the simple roots is Dynkin's, as used in [22].) The Weyl vector is $\rho=(1,1,\ldots,1)$. The colabels $a_i^\vee$ are defined through the expansion of the highest root ψ in the basis of simple roots, $\psi=\sum_i(\frac{2a_i^\vee}{\alpha_i^2})\alpha_i$. Put $a_0^\vee=1$. The dual Coxeter number $h^\vee = 1+\rho\cdot\psi=\sum_i a_i^\vee$.

By $X_{\ell,k}$ we will mean the current algebra based on X_ℓ, at a level $k\in\mathbb{N}$. The height is defined by $n=k+h^\vee$. The integrable highest weight representations of $X_{\ell,k}$ are in one–to–one correspondence with the set of dominant weights (also called the alcôve) of $X_{\ell,k}$ given by [22]

$$P_+(X_{\ell,k}) = \big\{\lambda=(\lambda_0;\lambda_1,\lambda_2,\ldots,\lambda_\ell)\mid \lambda_i\in\mathbb{N} \text{ and } \sum_{i=0}^{\ell} a_i^\vee\lambda_i = k\big\}, \tag{3.1}$$

and have characters denoted by $\chi_\lambda(\tau,z,u)$. For fixed level k, the zero–th Dynkin label is redundant, so that two notations $(\lambda_0;\lambda_1,\lambda_2,\ldots,\lambda_\ell)$ and $(\lambda_1,\lambda_2,\ldots,\lambda_\ell)$ designate a single element of $P_+(X_{\ell,k})$. The identity 0 corresponds to $k\omega^0 := (k;0,\ldots,0)$.

The characters $\{\chi_\lambda\}_{\lambda\in P_+(X_{\ell,k})}$ transform linearly under the action of $SL(2,\mathbb{Z})$, defined as follows by its generators $(\tau,z,u)\mapsto(\tau+1,z,u)$ and $(\tau,z,u)\mapsto(\frac{-1}{\tau},\frac{z}{\tau},u+\frac{z^2}{2\tau})$ [23]; these representing matrices (called Kac-Peterson matrices) are respectively

$$T_{\lambda,\lambda'} = \gamma\exp\left(\frac{2\pi i(\rho+\lambda)^2}{2n}\right)\delta_{\lambda,\lambda'}, \tag{3.2a}$$

$$S_{\lambda,\lambda'} = \gamma'\sum_{w\in W}(\det w)\exp\left(-\frac{2\pi i(\rho+\lambda)\cdot w(\rho+\lambda')}{n}\right). \tag{3.2b}$$

γ and γ' are constants independent of λ and λ', and W is the Weyl group of X_ℓ. The matrices S and T are both symmetric and unitary, and satisfy $S^2 = (ST)^3 = C$. C, called the charge conjugation, is an order 2 symmetry of the Dynkin diagram of X_ℓ (if non–trivial).

In particular the matrix elements $S_{0,\lambda}$ are all real and strictly positive, and obey $S_{0,\lambda} \geq S_{0,0}$ for all $\lambda \in P_+(X_{\ell,k})$. The quantum dimension $\mathcal{D}(\lambda)$ is defined by

$$\mathcal{D}(\lambda) := \frac{S_{0,\lambda}}{S_{0,0}} = \prod_{\alpha>0} \frac{\sin\left[\pi\alpha\cdot(\rho+\lambda)/n\right]}{\sin\left[\pi\alpha\cdot\rho/n\right]}, \tag{3.3}$$

where the product is over the positive roots of X_ℓ, and the second equality follows from the Weyl denominator formula. Note that $\mathcal{D}(\lambda) \geq 1$. Those weights λ which satisfy $\mathcal{D}(\lambda) = 1$ are called *simple currents* [31]. Except for one single case, namely $E_{8,2}$, they are all given [9] by the action $\lambda = J(k\omega^0)$ of a symmetry J of the extended Dynkin diagram which does not fix the zero–th node; these J act on weights by permuting their Dynkin labels. By abuse of language, the same notation J is used to denote the simple current and the corresponding permutation. The simple currents permute the weights in the alcôve and have the important property that

$$S_{\lambda,\lambda'} = \exp\left[-2\pi i Q_J(\lambda')\right] S_{J\lambda,\lambda'} = \exp\left[-2\pi i Q_J(\lambda)\right] S_{\lambda,J\lambda'}, \tag{3.4}$$

where the charge $Q_J(\lambda)$ and conformal weight h_J are defined by

$$Q_J(\lambda) \equiv h_\lambda + h_{J(0)} - h_{J(\lambda)} \mod 1, \tag{3.5a}$$

$$h_\lambda \equiv \frac{(\rho + J(\lambda))^2 - \rho^2}{2n} \mod 1. \tag{3.5b}$$

The simple currents were classified in [9] for all simple Lie algebras. Their explicit form will be given in the text (see sections 5 and 7).

The weights in $P_+(X_{\ell,k})$ form a ring, called the fusion ring: the elements are *formal* linear combinations over $\mathbb{Z}$ of the weights, and the product $\lambda \times \mu = \sum N^\nu_{\lambda,\mu}\nu$ has non–negative integer structure constants $N^\nu_{\lambda,\mu}$ called fusion coefficients. These are defined by the Verlinde formula [34]

$$N^\nu_{\lambda,\mu} = \sum_{\beta\in P_+(X_{\ell,k})} \frac{S_{\lambda,\beta}S_{\mu,\beta}S^*_{\nu,\beta}}{S_{0,\beta}}. \tag{3.6a}$$

They can be computed, at least in principle, by Lie algebraic methods. For example [22,36,14],

$$N^\nu_{\lambda,\mu} = \sum_{w\in\widehat{W}} (\det\, w)\, R^{w.\nu}_{\lambda,\mu} \tag{3.6b}$$

$$= \sum_{\beta\in P(\mu)} \sum_{\substack{w\in\widehat{W} \\ w.(\lambda+\beta)=\nu}} (\det\, w)\,\mathrm{mult}_\mu(\beta), \tag{3.6c}$$

where $w.\nu = w(\nu+\rho)-\rho$, $\widehat{W}$ is the (affine) Weyl group of $X_{\ell,k}$, and $R^\nu_{\lambda,\mu} \in \mathbb{N}$ are the Clebsch–Gordan series coefficients of the X_ℓ tensor product $\lambda\otimes\mu$. In (3.6c), $P(\mu)$ is the set of weights of the X_ℓ representation μ, and $\mathrm{mult}_\mu(\beta)$ is the multiplicity of β in μ. It should always be clear from the context whether "$\lambda+\mu$" refers to the formal sum of the fusion ring, or the usual component–wise sum.

In this paper, we will classify all modular invariant partition functions

$$Z = \sum_{\mu,\mu'\in P_+(X_{\ell,k})} M_{\mu,\mu'}\chi^*_\mu\,\chi_{\mu'} \tag{3.7}$$

for which the integer matrix M defines a permutation σ of the alcôve by $M_{\mu,\mu'} = \delta_{\mu',\sigma(\mu)}$. Modular invariance of (3.7) is equivalent to the statement that σ commutes with the matrices S and T, that is,

$$T_{\lambda,\lambda'} = T_{\sigma(\lambda),\sigma(\lambda')}, \tag{3.8a}$$

$$S_{\lambda,\lambda'} = S_{\sigma(\lambda),\sigma(\lambda')}. \tag{3.8b}$$

Any permutation σ of $X_{\ell,k}$ obeying (3.8a),(3.8b) is called an automorphism invariant. Note that they form a group under composition. Since the 0–th row of S is the only positive one, (3.8b) implies σ will fix the identity,

$$\sigma(0) = 0. \tag{3.8c}$$

From (3.6a) and (3.8b), σ is an automorphism of the fusion ring:

$$N_{\sigma(\lambda),\sigma(\mu)}^{\sigma(\nu)} = N_{\lambda,\mu}^{\nu} \tag{3.8d}$$

(the converse is not true though). For this reason, the corresponding partition functions are called permutation invariants or automorphism invariants.

We will denote the trivial permutation by σ_1. At present three main methods of systematically constructing non–trivial automorphism invariants are known: conjugations, simple currents and Galois transformations can be used. (In principle, these constructions are independent, but they sometimes overlap, as has recently been discussed [12].) Any symmetry of the Dynkin diagram which fixes the zero–th node is called a conjugation; they act on weights by permuting their Dynkin labels, and as such always define automorphism invariants.

Simple currents provide a large stock of automorphism invariants [31,20]. Let N be the order of a simple current J. When $Nh_{J(0)}$ is an integer coprime with N, we can define a simple current automorphism invariant by setting [32]

$$\sigma_J(\lambda) = J^a(\lambda), \qquad \text{with } ah_{J(0)} \equiv Q_J(\lambda) \bmod 1\ . \tag{3.9}$$

It can be checked that σ_J indeed commutes with T and S, and is a permutation of the alcôve.

Incidentally, many special cases of (3.9) were written down first by [3,1,13]. Let us also mention that, when two independent simple currents exist, a different kind of simple current automorphism than (3.9) sometimes exists, called an integer spin simple current automorphism [30]. For simple X_ℓ, this kind of automorphism only exists for the D_ℓ series (it is denoted by σ_{vsc} in Table 1), so that we refrain from giving the general description and merely refer to the D_ℓ–subsection 5.4 for its precise definition.

By $[\lambda]$ we mean the orbit of λ under all the conjugations C_i and simple currents J_j. These orbits play an important role in this paper.

Another way to construct modular invariants is by Galois transformations. We see from (3.2b) (in fact this holds for any RCFT [7]) that the matrix elements $S_{\lambda,\lambda'}$ lie in a cyclotomic extension of the rationals $\mathbb{Q}(\zeta_N) = \mathbb{Q}(\exp 2\pi i/N)$, for some algebra-dependent integer N. Its Galois group is isomorphic to $\mathrm{Gal}(\mathbb{Q}(\zeta_N)/\mathbb{Q}) \cong \mathbb{Z}_N^*$, the group of invertible integers modulo N. It is immediate from (3.2b) that any element g of the Galois group induces a permutation $\lambda \mapsto g(\lambda)$ of the alcôve through its action on S

$$g(S_{\lambda,\lambda'}) = \epsilon_g(\lambda) S_{g(\lambda),\lambda'} = \epsilon_g(\lambda') S_{\lambda,g(\lambda')}, \tag{3.10}$$

where $\epsilon_g(\lambda) = \pm 1$ is a sign that only depends on g and λ. The images $g(\lambda)$ and $g(\lambda')$, the Galois transforms of λ and λ', can be quite explicitly computed in the following way. Let g_a with $a \in \mathbb{Z}_N^*$ be a Galois transformation. Then $g_a(\lambda)$ is the unique weight in the alcôve such that $\rho + g_a(\lambda) = w_{a,\lambda}\big(a(\rho+\lambda)\big) + n\alpha_{a,\lambda}^\vee$ for some Weyl transformation $w_{a,\lambda}$ and some $\alpha_{a,\lambda}^\vee \in Q^\vee$, the co-root lattice. Also the sign appearing in (3.10) is given by $\epsilon_{g_a}(\lambda) = \det w_{a,\lambda}$.

Under some conditions, Galois transformations directly define automorphism invariants by setting $M_{\lambda,\lambda'} = \delta_{\lambda',g(\lambda)}$. This is the case whenever g fixes the identity, $g(0) = 0$, and commutes with T [11].

More generally, suppose that g is such that $g(0) = J(0)$ for some simple current J and that g commutes with T, and $g^2 = 1$. Then the following defines an automorphism invariant:

$$\sigma_g(\lambda) = \begin{cases} J(g(\lambda)) & \text{if } \epsilon_g(\lambda) = \epsilon_g(0), \\ g(\lambda) & \text{if } \epsilon_g(\lambda) = -\epsilon_g(0). \end{cases} \tag{3.11}$$

The proof is simple. The extreme l.h.s. and r.h.s. of the equalities

$$\exp[2\pi i Q_J(\lambda)] S_{0,\lambda} = S_{J(0),\lambda} = S_{g(0),\lambda} = \epsilon_g(0)\, \epsilon_g(\lambda)\, S_{0,g(\lambda)}, \tag{3.12}$$

imply

$$\exp[2\pi i Q_J(\lambda)] = \epsilon_g(0)\, \epsilon_g(\lambda). \tag{3.13}$$

The same equation (3.12) with λ replaced by $J(\lambda)$ shows that $Q_J(J(\lambda)) \equiv Q_J(\lambda)$ mod 1. Thus $Q_J(\lambda)$ can only take the values 0 and $\frac{1}{2}$ modulo 1, and J is a simple current of order 2, $J^2 = id$. Moreover acting with $g^2 = 1$ on $S_{0,\lambda}$, one obtains $Q_J(g(\lambda)) \equiv Q_J(\lambda)$ mod 1. That σ_g obeys (3.8a,b) can now be verified.

Equation (3.11) appears to be new. We will call the corresponding σ *generalized Galois automorphisms* since they reduce to pure Galois automorphisms if $g(0) = 0$. In section 6.3 we will show that the $B_{\ell,2}$ and $D_{\ell,2}$ exceptional invariants have precisely this form. Incidentally, in most cases (including all cases concerning us in this paper) $[g, T] = 0$ implies $g^2 = 1$. Moreover, the charge conjugation $C = S^2$ always corresponds to g_{-1}.

We finish this section with a lemma which will be repeatedly used throughout the paper. It is a slight generalization of a result proved in [17].

Lemma 1. *Let σ be an automorphism invariant for $X_{\ell,k}$. If σ fixes all $\omega^i \in P_+(X_{\ell,k})$, then σ is the trivial permutation on $P_+(X_{\ell,k})$.*

To prove this, it suffices to show that any $S_{\lambda,\mu}/S_{0,\mu}$ can be written as a polynomial P'_λ in the ratios $S_{\omega^i,\mu}/S_{0,\mu}$ for all $\omega^i \in P_+(X_{\ell,k})$. This is true because from (3.8b) and the fact that the identity 0 and all ω^i are fixed by σ, one obtains $S_{\lambda,\sigma(\mu)} = S_{\lambda,\mu}$ for all λ, μ, so that if $\sigma \neq 1$, two columns of S would be equal and S would be singular. We do know from [17] that $S_{\lambda,\mu}/S_{0,\mu}$ can be written as a polynomial P_λ in the ratios $S_{\omega^i,\mu}/S_{0,\mu}$ for all $1 \leq i \leq \ell$. The problem is that if k is small, not all ω^i may lie in $P_+(X_{\ell,k})$ (we use (3.2b) to extend the definition of $S_{\lambda,\mu}$ outside the alcôve).

Suppose $\nu \notin P_+(X_{\ell,k})$, for some weight ν. Then either ν lies in a wall, in which case

$$S_{\nu,\mu} = 0, \qquad \forall \mu \in P_+(X_{\ell,k}), \tag{3.14a}$$

or there exists a $\beta_\nu \in P_+(X_{\ell,k})$, a $w_\nu \in W$, and an element $\alpha^\vee_\nu$ of the co–root lattice such that $\rho + \nu = w_\nu(\rho + \beta_\nu) + n\alpha^\vee_\nu$, in which case

$$S_{\nu,\mu} = (\det w_\nu)\, S_{\beta_\nu,\mu}, \qquad \forall \mu \in P_+(X_{\ell,k}). \tag{3.14b}$$

All that we need to verify is that whenever $a^\vee_i > k$, either $\nu = \omega^i$ satisfies (3.14a), or it satisfies (3.14b) with $\beta_\nu = 0$ or ω^j for some j. This is automatic whenever $a^\vee_i = k + 1$, or when $P_+(X_{\ell,k})$ contains only weights of the form 0 and ω^j.

This leaves only $E_{7,2}$ with $\nu = \omega^3$, and $E_{8,4}$ with $\nu = \omega^5$. It suffices to show that $\beta_\nu \neq 2\omega^6$, respectively $\omega^1 + \omega^7$, $2\omega^1$ or $2\omega^7$. But $\rho + \nu$ and $\rho + \beta_\nu$ must have the same norm modulo $2n$, and checking the norms, we find that β_ν cannot take these values, so that Lemma 1 is proved for all k.

4. Quantum Dimensions

In this section we use quantum dimensions to find a weight ω^f at each level which must be fixed (up to extended Dynkin diagram symmetries) by any automorphism invariant σ.

Recall the definition of quantum dimension $\mathcal{D}(\lambda)$, given in (3.3). The positive roots α are explicitly given in e.g. [4]. Let $\mathcal{Q}_1$ be the set of all weights $\lambda \in P_+(X_{\ell,k})$ with the smallest value of $\mathcal{D}(\lambda)$, let $\mathcal{Q}_2$ be those with the second smallest value, etc. We know that for all $\lambda' \in [\lambda]$, $\mathcal{D}(\lambda) = \mathcal{D}(\lambda')$.

By (3.8b),(3.8c), we find that $\mathcal{D}(\lambda) = \mathcal{D}(\sigma\lambda)$, hence

$$\sigma \mathcal{Q}_m = \mathcal{Q}_m, \qquad \forall m = 1, 2, \ldots \tag{4.1a}$$

Fuchs [9] found the set $\mathcal{Q}_1$ for any $X_{\ell,k}$: in all cases except one, $\mathcal{Q}_1 = [0]$; the only exception is $E_{8,2}$, where $\mathcal{Q}_1 = [0] \cup [\omega^7]$. He proved this by regarding $\mathcal{D}(\lambda)$ as an analytic function of ℓ real variables $\lambda_1, \ldots, \lambda_\ell$, defined by the expression in (3.3). These λ are to lie in the convex hull

$$\overline{P}_+(X_{\ell,k}) := \Big\{ \sum_{i=0}^{\ell} \lambda_i \omega^i \mid \lambda_i \in \mathbb{R}_{\geq} \text{ and } \sum_{i=0}^{\ell} \lambda_i a_i^\vee = k \Big\}.$$

It was found in [9] that, for all $i = 1, \ldots, \ell$,

$$\frac{\partial}{\partial \lambda_i} \mathcal{D}(\lambda) = 0 \qquad \Longrightarrow \qquad \frac{\partial^2}{\partial \lambda_i \partial \lambda_j} \mathcal{D}(\lambda) < 0, \qquad \forall j = 1, \ldots, \ell. \tag{4.1b}$$

Though (4.1b) is not strong enough for our purposes, this basic idea will be a critical step in our analysis.

The main result of this section is the determination of $\mathcal{Q}_2$ for all $X_{\ell,k}$.

Proposition. (a) *for $X_\ell = A_\ell$, $\ell \geq 1$ and $k \geq 2$: $\mathcal{Q}_2 = [\omega^1]$,*

(b) *for $X_\ell = B_\ell$, $\ell \geq 3$ and $k \geq 4$: $\mathcal{Q}_2 = [\omega^1]$,*

(c) *for $X_\ell = C_\ell$, $\ell \geq 2$ and $k = 1$ or $\ell + k \geq 6$: $\mathcal{Q}_2 = [\omega^1]$,*

(d) *for $X_\ell = D_{\ell \geq 4}$ and E_6, $k \geq 3$: $\mathcal{Q}_2 = [\omega^1]$,*

(e) *for $X_\ell = E_7$, E_8, F_4 and G_2, $k \geq 5$: $\mathcal{Q}_2 = [\omega^6]$, $[\omega^1]$, $[\omega^4]$ and $[\omega^2]$ respectively.*

For the levels missed by the Proposition, we have listed in Table 3 the sets $\mathcal{Q}_m$ for small m. Together with the T–condition (3.8a) and the selection rule (4.1a), the Proposition and Table 3 give us the following valuable facts.

Corollary. *An automorphism invariant σ necessarily satisfies:*

(a) *$\sigma\omega^1 \in [\omega^1]$ for A_ℓ, C_ℓ or E_6, any k,*

(b) *$\sigma\omega^6 \in [\omega^6]$ for E_7, any k,*

(c) *$\sigma\omega^1 \in [\omega^1]$ for B_ℓ and D_ℓ, any $k \neq 2$,*

(d) *$\sigma\omega^4 = \omega^4$ for F_4, any $k \neq 3$,*

(e) *$\sigma\omega^1 = \omega^1$ and $\sigma\omega^2 = \omega^2$ for E_8 and G_2 respectively, any $k \neq 4$.*

In what follows, we will denote by ω^f the weight singled out by the Proposition and Corollary — so $\omega^f = \omega^1$ for all but E_7, F_4, G_2. Note that in all cases ω^f is the weight of X_ℓ with second smallest Weyl dimension. This is of course not a coincidence, and happens because, for fixed λ, $\lim_{k\to\infty}\mathcal{D}(\lambda)$ is the Weyl dimension of λ.

The remainder of this section is devoted to the proof of the Proposition.

X_ℓ	k	$\mathcal{Q}_1$	$\mathcal{Q}_2$	$\mathcal{Q}_3$	$\mathcal{Q}_4$	$\mathcal{Q}_5$
A_ℓ	1	$[0]=[\omega^1]$				
B_ℓ	1	$[0]=[\omega^1]$				
	2	$[0]$	$[\omega^1]\cup\cdots\cup[\omega^{\ell-1}]\cup[2\omega^\ell]$			
	3	$[0]$	$[3\omega^\ell]$	$[\omega^1]$		
C_2	2	$[0]$	$[\omega^2]\cup[2\omega^1]$	$[\omega^1]$		
C_2	3	$[0]$	$[\omega^1]\cup[\omega^2]\cup[3\omega^1]$			
C_3	2	$[0]$	$[\omega^1]\cup[\omega^3]\cup[2\omega^1]$			
D_ℓ	1	$[0]=[\omega^1]$				
D_4	2	$[0]$	$[\omega^1]\cup\cdots\cup[\omega^\ell]$			
$D_{\ell>4}$	2	$[0]$	$[\omega^1]\cup\cdots\cup[\omega^{\ell-1}]$			
E_6	1	$[0]=[\omega^1]$				
	2	$[0]$	$[\omega^2]$	$[\omega^1]$		
E_7	1	$[0]=[\omega^6]$				
	2	$[0]$	$[\omega^7]$	$[\omega^1]$	$[\omega^6]$	
	3	$[0]$	$[\omega^1]\cup[\omega^2]\cup[\omega^6]$			
	4	$[0]$	$[2\omega^7]$	$[\omega^6]$		
E_8	1	$[0]$				
	2	$[0]\cup[\omega^7]$	$[\omega^1]$			
	3	$[0]$	$[\omega^8]$	$[\omega^2]$	$[\omega^1]$	
	4	$[0]$	$[2\omega^7]$	$[2\omega^1]$	$[\omega^1]\cup[\omega^6]$	
F_4	1	$[0]$	$[\omega^4]$			
	2	$[0]$	$[\omega^1]$	$[2\omega^4]$	$[\omega^3]$	$[\omega^4]$
	3	$[0]$	$[\omega^1]\cup[3\omega^4]$	$[\omega^2]\cup[\omega^4]$		
	4	$[0]$	$[\omega^1]\cup[\omega^4]\cup[2\omega^1]\cup[4\omega^4]$			
G_2	1	$[0]$	$[\omega^2]$			
	2	$[0]$	$[\omega^1]$	$[2\omega^2]$	$[\omega^2]$	
	3	$[0]$	$[\omega^1]\cup[\omega^2]\cup[3\omega^2]$			
	4	$[0]$	$[\omega^2]\cup[2\omega^1]$			

Table 3. Quantum dimensions for small k. Here are listed those exceptional cases missing in the Proposition, and the order on the orbits of the weights, up to $[\omega^f]$, induced by their quantum dimensions.

Step 1. The first step in the proof of the Proposition will be to analyse (3.3), in order to come up with a small list of candidates $\lambda \in P_+(X_{\ell,k})$ for belonging to $\mathcal{Q}_2$.

Choose any constants $a=\sum_{i=0}^{\ell} a_i\omega^i$, $b=\sum_{i=0}^{\ell} b_i\omega^i \neq 0$, $a_i, b_i \in \mathbb{R}$. Suppose that for all $t\in[t_0,t_1]$,

$a + bt \in \overline{P}_+(X_{\ell,k})$. Then for $t_0 \le t' \le t_1$, an easy calculation gives

$$\frac{d}{dt}\mathcal{D}(a+bt)\big|_{t=t'} = 0 \qquad \Longrightarrow \qquad (4.2a)$$

$$\frac{d^2}{dt^2}\mathcal{D}(a+bt)\big|_{t=t'} = -\mathcal{D}(a+bt')\,\frac{\pi^2}{n^2}\sum_{\alpha>0}\frac{(b\cdot\alpha)^2}{\sin^2[\pi\,(a+bt'+\rho)\cdot\alpha/n]} < 0.$$

This means that $\mathcal{D}(a+b\,t)$ will attain its minimum at one of the endpoints $t = t_0, t_1$:

$$\text{for all } t_0 < t' < t_1, \quad \mathcal{D}(a+b\,t') > \min\{\mathcal{D}(a+b\,t_0),\, \mathcal{D}(a+bt_1)\}. \qquad (4.2b)$$

This implies the following rule. Suppose

$$\sum_{i=0}^{\ell} m_i a_i^\vee = 0, \qquad m_i \in \mathbb{Z}, \qquad (4.2c)$$

and not all $m_i = 0$. If $\lambda \in \mathcal{Q}_2$ and both $\lambda \pm m \notin \mathcal{Q}_1$, then

$$\lambda_i < |m_i| \quad \text{for some } 0 \le i \le \ell. \qquad (4.2d)$$

To prove this, we take a weight λ of $P_+(X_{\ell,k})$ and consider the family $\lambda(t) = \lambda + mt$. With $t_0 = \max_{i\,:\,m_i>0}(-\lambda_i/m_i)$ and $t_1 = \min_{i\,:\,m_i<0}(-\lambda_i/m_i)$, the weights $\lambda(t)$ belong to $\overline{P}_+(X_{\ell,k})$ for all $t \in [t_0,t_1]$. Note that $\lambda(t)$ will belong to $P_+(X_{\ell,k})$ if $t \in [t_0,t_1]$ is integer. If both $\pm 1 \in [t_0,t_1]$, one obtains from (4.2b) a contradiction to $\lambda \in \mathcal{Q}_2$ unless $\lambda(\pm 1) \in \mathcal{Q}_1$. Thus unless one of $\lambda \pm m \in \mathcal{Q}_1$, we must have $t_0 > -1$ or $t_1 < 1$, implying (4.2d).

Writing $a_{ij} = \gcd(a_i^\vee, a_j^\vee)$, a special case of (4.2d) is that, for each choice of $0 \le i < j \le \ell$,

$$\text{either} \quad \lambda_i < \frac{a_j^\vee}{a_{ij}} \quad \text{or} \quad \lambda_j < \frac{a_i^\vee}{a_{ij}}, \qquad (4.2e)$$

again provided $\lambda \pm (\frac{a_j^\vee}{a_{ij}}\omega^i - \frac{a_i^\vee}{a_{ij}}\omega^j) \notin \mathcal{Q}_1$. Eq.(4.2e) implies that if $\lambda \in \mathcal{Q}_2$, then at most one λ_i can be larger than $\max_j a_j^\vee$.

Any λ which obeys (4.2d) for all choices of m_i satisfying eq.(4.2c), will be called a *candidate*. Step 1 consists of finding all candidates. The result is given in the lemma below, where we use the following notation. Define the *truncation* $[c]$ to be the largest integer not greater than c, and the *remainder* $\{c\}_d$ to be $c - d[c/d]$. By $\mu(ij)$ we mean the weight

$$\mu(ij) := \{k\}_{a_{ij}}\omega^0 + x\omega^i + y\omega^j, \qquad (4.3a)$$

where x and y are given by

$$x = \left\{[k/a_{ij}](a_i')^{-1}\right\}_{a_j'}, \qquad y = \frac{k - \{k\}_{a_{ij}} - a_i^\vee x}{a_j^\vee}. \qquad (4.3b)$$

In (4.3b), $a_i' = a_i^\vee/a_{ij}$ and $a_j' = a_j^\vee/a_{ij}$, and by $(a_i')^{-1}$ we mean the (integer) multiplicative inverse mod a_j'. For example, if $a_i^\vee = 2$ and $a_j^\vee = 3$, we get $(x,y) = (0,\frac{k}{3})$, $(2,\frac{k-4}{3})$, $(1,\frac{k-2}{3})$ for $k \equiv 0,1,2 \bmod 3$, respectively, while if $a_i^\vee = 3$ and $a_j^\vee = 2$ we get $(x,y) = (0,\frac{k}{2})$, $(1,\frac{k-3}{2})$ for $k \equiv 0,1 \bmod 2$, respectively. Note that $\mu(ij) = \mu(0j)$ if $a_j^\vee$ divides either $a_i^\vee$ or k.

The virtue of (4.3a) is that it gives in one formula almost all candidates which have at most three non–zero Dynkin labels, one of them being λ_0 if it has exactly three non–zero labels. Suppose for instance $\lambda_0, \lambda_i, \lambda_j \ne 0$. Then from (4.2d), the choices $(m_0,m_i) = (a_i^\vee,-1)$, $(m_0,m_j) = (a_j^\vee,-1)$ and $(m_0,m_i,m_j) = (a_i^\vee - a_j^\vee, -1, 1)$ (all others zero in each case) lead to $\lambda_0 < \min\{a_i^\vee, a_j^\vee, |a_i^\vee - a_j^\vee|\}$. Checking all possible pairs

of colabels, one can see that it implies $\lambda_0 < a_{ij}$, except for E_8 if $\{a_i^\vee, a_j^\vee\} = \{2,5\}$ or $\{3,5\}$. Moreover (4.2e) implies $\lambda_i < a'_j$ or $\lambda_j < a'_i$, say $\lambda_i < a'_j$ for definiteness. Then

$$\lambda_0 + a_i^\vee \lambda_i + a_j^\vee \lambda_j = k \quad \Longrightarrow \quad \begin{cases} \lambda_0 \equiv k \bmod a_{ij}, \\ \lambda_i \equiv \frac{k-\lambda_0}{a_{ij}} (a'_i)^{-1} \bmod a'_j. \end{cases} \tag{4.3c}$$

When $\lambda_0 < a_{ij}$, the r.h.s. of (4.3c) uniquely fixes λ_0 and λ_i, which then determines the value of λ_j using the l.h.s. of (4.3c) – that is, in this case we find that indeed $\lambda = \mu(ij)$. Finally for $E_{8,k}$, $k \equiv 3, 4 \bmod 5$, there are four candidates with $\lambda_0 \geq a_{ij}$, given separately in Lemma 2.

Lemma 2. (1) *The candidates for $A_{\ell,k}$ are $[\omega^i]$, $1 \leq i \leq \frac{\ell+1}{2}$.*

(2) *The candidates for $B_{\ell,k}$ are : $[\omega^i]$ for $1 \leq i \leq \ell$, $[\mu(0j)]$ and $[\mu(\ell j)]$ for $1 < j < \ell$, and $[\mu(0\ell)]$.*

(3) *The candidates for $C_{\ell,k}$ are : $[\omega^i]$ for $1 \leq i \leq \ell$ and $[\mu(0j)]$ for $1 \leq j \leq \ell/2$.*

(4) *The candidates for $D_{\ell,k}$ are : $[\omega^i]$ for $1 \leq i < \ell$ and $[\mu(0j)]$ for $1 < j < \ell - 1$.*

(5) *The candidates for $E_{6,k}$ are : $[\omega^i]$ for $i = 1, 2, 3, 6$, $[\mu(0j)]$ for $j = 2, 3, 6$, $[\mu(23)]$ and $[\mu(32)]$.*

(6) *The candidates for $E_{7,k}$ are : $[\omega^i]$ for $1 \leq i \leq 7$, $[\mu(ij)]$ for most pairs $0 \leq i \neq j \leq 7$, and $[\omega^{1,7} + \omega^{2,4} + \frac{k-5}{4}\omega^3]$ for $k \equiv 1 \bmod 4$.*

(7) *The candidates for $E_{8,k}$ are : $[\omega^i]$ for $1 \leq i \leq 8$, $[\mu(ij)]$ for most pairs $0 \leq i \neq j \leq 8$, as well as*
for $k \equiv 1 \bmod 4$, $\omega^{1,7} + \omega^{2,8} + \frac{k-5}{4}\omega^{3,6}$ *and* $\omega^{2,8} + \frac{k-9}{4}\omega^{3,6} + \omega^5$,
for $k \equiv 3 \bmod 4$, $\omega^{1,7} + \frac{k-7}{4}\omega^{3,6} + \omega^4$ *and* $\frac{k-11}{4}\omega^{3,6} + \omega^4 + \omega^5$,
for $k \equiv 1 \bmod 5$, $\omega^{1,7} + \omega^{3,6} + \frac{k-6}{5}\omega^4$,
for $k \equiv 2 \bmod 5$, $\omega^{2,8} + \omega^{3,6} + \frac{k-7}{5}\omega^4$,
for $k \equiv 3 \bmod 5$, $\omega^{1,7} + \frac{k-3}{5}\omega^4$, *and* $\omega^{1,7} + \frac{k-8}{5}\omega^4 + \omega^5$,
for $k \equiv 4 \bmod 5$, $\omega^{2,8} + \frac{k-4}{5}\omega^4$ *and* $\omega^{2,8} + \frac{k-9}{5}\omega^4 + \omega^5$,
for $k \equiv 1, 2 \bmod 6$, $\omega^{2,8} + (3 - \{k\}_6)\omega^4 + [\frac{k-8}{6}]\omega^5$,
for $k \equiv 1, 3 \bmod 6$, $\frac{1+\{k\}_6}{2}\omega^{1,7} + \omega^4 + [\frac{k-6}{6}]\omega^5$ *and* $\frac{5-\{k\}_6}{2}\omega^{3,6} + \omega^4 + [\frac{k-9}{6}]\omega^5$,
for $k \equiv 1, 5 \bmod 6$, $\frac{9-\{k\}_6}{4}\omega^{1,7} + \omega^{2,8} + [\frac{k-5}{6}]\omega^5$ *and* $\omega^{2,8} + \frac{3+\{k\}_6}{4}\omega^{3,6} + [\frac{k-6}{6}]\omega^5$,
for $k = 15$, $2\omega^{2,8} + \omega^{3,6} + \omega^4$.

(8) *The candidates for $F_{4,k}$ are : ω^i for $i = 1, 2, 3, 4$, $\mu(0j)$ for $j = 1, 2, 3, 4$, $\mu(4j)$ for $j = 1, 2, 3$, and $\mu(12)$, $\mu(21)$, $\mu(23)$, $\mu(32)$.*

(9) *The candidates for $G_{2,k}$ are : ω^1, ω^2, $\mu(01)$, $\mu(02)$ and $\mu(21)$.*

We use the notation $\omega^{1,7}$, etc, to denote *either* ω^1 *or* ω^7 (but not both simultaneously). Lemma 2 holds for any k, though for small k not all of these weights will lie in $P_+(X_{\ell,k})$. Also, for some k these candidates will not all be distinct: e.g. for $B_{\ell,k}$, k even, $\mu(\ell j) = \mu(0j)$.

We will sketch the proof for the hardest case, namely E_8. First note that by (4.2d), at most one element in each of the pairs (λ_1, λ_7), (λ_2, λ_8), (λ_3, λ_6) can be different from zero. For notational convenience, suppose that $\lambda_6 = \lambda_7 = \lambda_8 = 0$.

Together with (4.2d), the seven arithmetic identities

$$0 = 1 + 2 - 3 = 1 + 3 - 4 = 2 + 3 - 5 = 1 + 5 - 6 = 1 + 4 - 5 = 2 + 4 - 6 = 3 - 4 - 5 + 6$$

tell us that at most three among $\lambda_0, \ldots, \lambda_5$ can be non–zero. If only one or two of the λ_i, for $i > 0$, are non–zero, then λ will equal either ω^i or $\mu(ij)$, or equal $\omega^1 + \frac{k-3}{5}\omega^4$ or $\omega^2 + \frac{k-4}{5}\omega^4$, as we have seen. Thus we may assume here that exactly three of λ_i are non–zero, and $\lambda_0 = \lambda_6 = \lambda_7 = \lambda_8 = 0$.

Now we just run through the various possibilities. For example, suppose $\lambda_1, \lambda_2, \lambda_3 \neq 0$. From (4.2e) we have $\lambda_1 = 1$. Since $-2 + 2 \cdot 3 - 4 = 0$, the inequality (4.2d) requires $\lambda_2 = 1$. Then the fact that λ

should be in the alcôve at all, fixes λ_3 and constrains k. For another example, suppose $\lambda_3, \lambda_4, \lambda_5 \neq 0$. Then $-4+2\cdot 5-6=0$, so that (4.2d) forces $\lambda_4 = 1$ and (4.2e) requires either $\lambda_3 \leq 2$ or $\lambda_5 = 1$.

Step 2. Here we will use rank–level duality of the quantum dimensions [28] to significantly reduce the numbers of candidates given in Lemma 2, for A_ℓ, B_ℓ, C_ℓ, D_ℓ.

There is a well–known duality between the quantum dimensions of $A_{\ell,k}$ and $A_{k-1,\ell+1}$, $C_{\ell,k}$ and $C_{k,\ell}$, and $SO(m)_k$ and $SO(k)_m$. In particular, writing $X_{\ell,k} \leftrightarrow X'_{k',\ell'}$, we have

$$2^a\,\mathcal{D}(\lambda) = \mathcal{D}'(\lambda'), \tag{4.4}$$

where $\mathcal{D}'$ is the quantum dimension for the dual theory $X'_{k',\ell'}$. In all cases, the weight 0 for $X_{\ell,k}$ is sent to the weight 0 for $X'_{k',\ell'}$, and $(\lambda')' \in [\lambda]$. For $X_\ell = A_\ell$ or C_ℓ, $a = 0$ and λ' is defined by saying its Young tableau is the transpose of that of λ (for this purpose we may identify C_1 with A_1). The situation for $X_\ell = B_\ell$ and D_ℓ is slightly more complicated; we will give below all relevant values of λ' and a.

When $X_\ell = B_\ell$ and $k > 6$, for each $1 \leq j < \ell$, $(\omega^j)' = j\omega'^1$ with $a = 0$. Also, for each $1 < j < \ell$, $([k/2]\omega^j)' = 2j\omega'^{k'}$ with $a = 0$ for k odd and $a = -1$ for k even. For k even, $(k\omega^\ell)' = 2\ell\omega'^{k'}$ with $a = -1$, while for k odd, $(k\omega^\ell)' = \omega'^{k'}$ with $a = -\frac{1}{2}$. Finally, when k is odd, $(\omega^\ell)' = (2\ell+1)\omega'^{k'}$ with $a = \frac{1}{2}$, and for each $1 < j < \ell$, $\mu(\ell j)' = (2\ell+1-2j)\omega'^{k'}$ with $a = -\frac{1}{2}$.

When $X_\ell = D_\ell$ and $k > 6$, for each $1 \leq j < \ell-1$, $(\omega^j)' = j\omega'^1$ with $a = 0$, and for $1 < j < \ell-1$, $([k/2]\omega^j)' = 2j\omega'^{k'}$ with $a = 0$ if k is odd, and $a = -1$ if k is even.

This rank-level duality for $X_\ell = B_\ell$ and D_ℓ extends to $3 \leq k \leq 6$ provided: we identify $B_{1,m}$ with $A_{1,2m}$ and put $\omega'^1 := 2\tilde{\omega}^1$, $\omega'^{k'} := \tilde{\omega}^1$; we identify $D_{2,m}$ with $A_{1,m} \oplus A_{1,m}$ and put $\omega'^1 := \tilde{\omega}^1 + \tilde{\omega}^2$, $\omega'^{k'} := \tilde{\omega}^1$; we identify $B_{2,m}$ with $C_{2,m}$ and put $\omega'^1 := \tilde{\omega}^2$, $\omega'^{k'} := \tilde{\omega}^1$; and we identify $D_{3,m}$ with $A_{3,m}$ and put $\omega'^1 := \tilde{\omega}^2$, $\omega'^{k'} := \tilde{\omega}^1$. By $\tilde{\omega}^i$ here we mean the fundamental weights for A_1, $A_1 \oplus A_1$, C_2, and A_3, respectively.

Now we turn to the consequences of this rank-level duality for finding $\mathcal{Q}_2$. Consider first $X_\ell = C_\ell$. Since the duality here between quantum dimensions is exact (*i.e.* $a = 0$ always), we have $\lambda \in \mathcal{Q}_2$ iff $\lambda' \in \mathcal{Q}'_2$. This gives us an additional constraint on $\lambda \in \mathcal{Q}_2$: λ' must be a candidate of $C_{k,\ell}$. However, $(\omega^j)' = j\omega'^1$ and $(k\omega^j)' = j\omega'^k$, so of these only ω^1, ω^ℓ and $k\omega^1$ are the duals of candidates. $X_\ell = A_\ell$ is similar.

The argument for $X_\ell = B_\ell$ and D_ℓ is not much more difficult. Consider for example B_ℓ when $k > 1$ is odd, and any $1 < j < \ell$:

$$\mathcal{D}(\omega^j) = \mathcal{D}'(j\omega'^1) > \min\{\mathcal{D}'(\omega'^1),\, \mathcal{D}'(2\ell\omega'^1)\} = \mathcal{D}'(\omega'^1) = \mathcal{D}(\omega^1), \tag{4.5a}$$

$$\begin{aligned}\mathcal{D}(\frac{k-1}{2}\omega^j) =& \mathcal{D}'(2j\omega'^{k'}) > \min\{\mathcal{D}'(\omega'^{k'}),\, \mathcal{D}'((2\ell+1)\omega'^{k'})\}\\ =& \min\{\mathcal{D}(\frac{k-1}{2}\omega^1),\, \sqrt{2}\mathcal{D}(\omega^\ell)\} \geq \min\{\mathcal{D}(\omega^1),\, \mathcal{D}(\omega^\ell)\},\end{aligned} \tag{4.5b}$$

$$\begin{aligned}\mathcal{D}(\frac{k-1}{2}\omega^j + \omega^\ell) = \sqrt{2}\,\mathcal{D}'((2\ell+1-2j)\,\omega'^{k'})\\ > \sqrt{2}\min\{\mathcal{D}'((2\ell+1)\omega'^{k'}),\, \mathcal{D}'(\omega'^{k'})\} \leq \min\{\mathcal{D}(k\omega^\ell),\, \mathcal{D}(\omega^\ell)\}.\end{aligned} \tag{4.5c}$$

In deriving (4.5) we use both rank–level duality and (4.2b).

Summarizing, we find the following results:

(1) for $A_{\ell,k}$ and $k \geq 2$: $\mathcal{Q}_2 = [\omega^1]$,
(2) for $B_{\ell,k}$ and $k \geq 3$: $\mathcal{Q}_2 \subseteq [\omega^1] \cup [\omega^\ell] \cup [k\omega^\ell]$,
(3) for $C_{\ell,k}$: $\mathcal{Q}_2 \subseteq [\omega^1] \cup [\omega^\ell] \cup [k\omega^1]$,
(4) for $D_{\ell,k}$ and $k \geq 3$: $\mathcal{Q}_2 \subseteq [\omega^1] \cup [\omega^\ell]$.

Step 3. The remaining candidates λ come in two forms. Some are independent of k (ignoring λ_0), while others have an index $j > 0$ for which the Dynkin label λ_j grows linearly with k. The quantum dimensions

of the first kind of candidates converge as $k\to\infty$ to the corresponding Weyl dimensions, while the quantum dimensions of the second kind of candidates will all tend to infinity. We will consider the two kinds of candidates separately; in this Step 3 we first address those independent of k. The quantum dimensions of the final four candidates in Lemma 2, all for $E_{8,15}$, can be explicitly computed, and are all found to be far larger than $\mathcal{D}_{15}(\omega^1)$. All other k-independent candidates are of the form ω^i. For the classical algebras, this step permits us to complete the proof of the Proposition.

Let λ,μ be independent of k, and lie in $P_+(X_{\ell,k_0})$. Then directly from (3.3) we find (a similar calculation was done in [10])

$$\frac{\partial}{\partial k}\frac{\mathcal{D}_k(\lambda)}{\mathcal{D}_k(\mu)}=\frac{\mathcal{D}_k(\lambda)}{\mathcal{D}_k(\mu)}\,E_k(\lambda+\rho,\mu+\rho)\ , \tag{4.6a}$$

where

$$E_k(\beta,\gamma):=\frac{\pi}{n^2}\sum_{\alpha>0}\Big[\gamma\cdot\alpha\,\cot\Big(\pi\,\frac{\gamma\cdot\alpha}{n}\Big)-\beta\cdot\alpha\,\cot\Big(\pi\,\frac{\beta\cdot\alpha}{n}\Big)\Big]\ . \tag{4.6b}$$

From (4.6a), we find that if $\mathcal{D}_{k_0}(\lambda)\geq\mathcal{D}_{k_0}(\mu)$, and $E_k(\lambda+\rho,\mu+\rho)>0$ for all $k\geq k_0$, then $\mathcal{D}_k(\lambda)>\mathcal{D}_k(\mu)$ for all levels $k>k_0$. Thus we begin by verifying the following, for all $k\geq 1$:

(i) for B_ℓ and $\ell\geq 4$: $E_k(\omega^\ell+\rho,\omega^1+\rho)>0$,
(ii) for C_ℓ and $\ell\geq 2$: $E_k(\omega^\ell+\rho,\omega^1+\rho)>0$,
(iii) for D_ℓ and $\ell\geq 5$: $E_k(\omega^\ell+\rho,\omega^1+\rho)>0$,
(iv) for E_6 : $E_k(\omega^i+\rho,\omega^1+\rho)>0$ for $i=2,3,6$,
(v) for E_7 : $E_k(\omega^i+\rho,\omega^6+\rho)>0$ for all $i\neq 6$,
(vi) for E_8 : $E_k(\omega^i+\rho,\omega^1+\rho)>0$ for all $i\neq 1$,
(vii) for F_4 : $E_k(\omega^i+\rho,\omega^4+\rho)>0$ for all $i\neq 4$,
(viii) for G_2 : $E_k(\omega^1+\rho,\omega^2+\rho)>0$.

That the result (iii) does not hold for $\ell=4$ is expected since $\omega^4\in[\omega^1]$ there, and is of no consequence. On the other hand, B_3 missing from (i) means it will have to be treated separately.

We will illustrate how to obtain (i)–(viii), by working out B_ℓ explicitly. Defining $c_i(x)=|\{\alpha>0\mid\alpha\cdot(\rho+\omega^i)=x\}|$, we find for B_ℓ:

$$c_1(x)-c_\ell(x)=\begin{cases}-2 & \text{if } x=1,\\ -1 & \text{if } 3\leq x\leq 2\ell-3 \text{ is an odd integer,}\\ 1 & \text{if } 2x\neq 2\ell-1 \text{ is an odd integer between 1 and } 2\ell+1,\\ 0 & \text{otherwise.}\end{cases} \tag{4.7}$$

Setting $f(x):=\frac{\pi x}{n^2}\cot\frac{\pi x}{n}$, we deduce from (4.7) that for all $k\geq 1$ and $\ell\geq 4$,

$$\begin{aligned}E_k(\omega^\ell+\rho,\omega^1+\rho)=&-f(1)+f(\ell+\frac{1}{2})+\sum_{j=0}^{\ell-2}\big\{f(j+\frac{1}{2})-f(2j+1)\big\}\\ =&\big\{f(\frac{5}{2})-f(3)\big\}-\big\{f(1)-f(\frac{3}{2})\big\}+\big\{f(\frac{1}{2})-f(1)\big\}\\ &+\sum_{j=3}^{\ell-2}\big\{f(j+\frac{1}{2})-f(2j-1)\big\}+\big\{f(\ell+\frac{1}{2})-f(2\ell-3)\big\}.\end{aligned} \tag{4.8}$$

The difference of the first two braces is strictly positive because the function $f(x)$ is concave over $[0,n[$, while the other terms are positive since $f(x)$ decreases over $[0,n[$. Thus $E_k(\omega^\ell+\rho,\omega^1+\rho)>0$ in this case. The other X_ℓ are done similarly.

Next, we will find a k_0 such that all $\omega^i\in P_+(X_{\ell,k_0})$, and $\mathcal{D}_{k_0}(\omega^f)\leq\mathcal{D}_{k_0}(\omega^i)$. For the exceptional algebras this is easy: we just explicitly compare the quantum dimensions $\mathcal{D}_k(\omega^i)$ for the small levels $k\geq\max_j\{a_j^\vee\}$. We find, for E_6, E_7, E_8, F_4, and G_2, that $k_0=3$, 4, 6, 4, and 3, respectively.

For $X_\ell = B_\ell$ ($\ell > 3$) and D_ℓ ($\ell > 4$), it suffices to note that at $k = 1$, $\omega^f \in [0]$ and $\omega^\ell \in P_+(X_{\ell,1})$. For $X_\ell = C_\ell$, it suffices to compute the level 2 quantum dimensions, which is easy to do from rank–level duality:

$$\frac{\mathcal{D}_2(\omega^\ell)}{\mathcal{D}_2(\omega^1)} = \left(\frac{1}{2\sin(\frac{\pi}{2\ell+6})\,\sin(\frac{3\pi}{2\ell+6})}\right) \Big/ \left(4\cos(\frac{\pi}{2\ell+6})\,\cos(\frac{3\pi}{2\ell+6})\right) = \frac{1}{2\sin(\frac{\pi}{\ell+3})\,\sin(\frac{3\pi}{\ell+3})}. \tag{4.9}$$

We find from (4.9) that $\mathcal{D}_2(\omega^\ell) \geq \mathcal{D}_2(\omega^1)$ for all $\ell \geq 3$, with equality only if $\ell = 3$. One more calculation then shows that for C_2, $\mathcal{D}_3(\omega^1) = \mathcal{D}_3(\omega^2)$.

Finally for B_3, the quantity $E_k(\omega^3 + \rho, \omega^1 + \rho)$, as given on the first line of (4.8), is negative for all $k \geq 1$, so that $\mathcal{D}_k(\omega^3)/\mathcal{D}_k(\omega^1)$ decreases with k. But since its value tends to $8/7$ as $k \to \infty$ (the ratio of the Weyl dimensions), it is bigger than 1 for all k.

Hence from (i)–(iii) above, together with the results of the previous step, one obtains that $[\omega^1]$ has the unique smallest quantum dimension among the $[\omega^i]$ for $B_{\ell,k}$ and $D_{\ell,k}$, $k \geq 3$, and also for $C_{\ell,k}$, $\ell, k \geq 2$ and $\ell + k \geq 6$.

This immediately concludes the proof of the Proposition for $D_{\ell,k}$, $k > 2$, but in fact is also enough to complete the proof for $B_{\ell,k}$ and $C_{\ell,k}$. For $C_{\ell,k}$, the rank–level duality described in the previous step implies

$$\mathcal{D}(k\omega^1) = \mathcal{D}'(\omega'^k) > \mathcal{D}'(\omega'^1) = \mathcal{D}(\omega^1), \tag{4.10}$$

for all $\ell, k \geq 2$ and $\ell + k \geq 6$. When $k = 1$, $k\omega^1 = \omega^1$ and $\omega^\ell \in [0]$. The remaining cases $C_{2,2}$, $C_{2,3}$ and $C_{3,2}$ can be checked explicitly with the results given in Table 3.

For B_ℓ, and $k > 6$ odd, rank–level duality implies

$$\mathcal{D}(k\omega^\ell) = \sqrt{2}\,\mathcal{D}'(\omega'^{(k-1)/2}) > \sqrt{2}\,\mathcal{D}'(\omega'^1) = \sqrt{2}\,\mathcal{D}(\omega^1) > \mathcal{D}(\omega^1). \tag{4.11}$$

The same applies when $k > 6$ is even. For $k = 3$, we can explicitly compute all quantum dimensions, using rank–level duality; we find the result indicated in Table 3. For $3 < k \leq 6$, it suffices to show $\mathcal{D}(\omega^1) < \mathcal{D}(k\omega^\ell)$ — again, rank–level duality is the most efficient means. For example,

$$\mathcal{D}_4(\omega^1) = \frac{\sin^2(\frac{2\pi}{2\ell+3})}{\sin^2(\frac{\pi}{2\ell+3})}, \qquad \mathcal{D}_4(4\omega^\ell) = 2\frac{\sin(\frac{2\pi}{2\ell+3})}{\sin(\frac{\pi}{2\ell+3})}. \tag{4.12}$$

Step 4. All that remains is to compare $\mathcal{D}_k(\omega^f)$ with $\mathcal{D}_k(\lambda^k)$ for those candidates λ^k of the exceptional algebras which depend explicitly on k. For each λ^k, there exists a unique Dynkin index $j > 0$ such that $(\lambda^k)_j$ grows like $k/a_j^\vee$. For each λ^k, we will consider $\mathcal{D}_k(\lambda^k)$ separately for each congruence class of k modulo $a_j^\vee$. Then $\mathcal{D}_k(\lambda^k)$ along such a congruence class can be written as a product of

$$g_{\alpha\beta\gamma}(n) := \frac{\sin(\pi\,(\alpha + \beta/n))}{\sin(\pi\,\gamma/n)}, \qquad . \tag{4.13a}$$

for α, β, γ independent of n and obeying the inequalities

$$0 \leq \alpha \leq \frac{1}{2}, \quad 0 < \alpha + \beta/n < 1, \quad 0 < \gamma < n. \tag{4.13b}$$

Now, $g_{\alpha\beta\gamma}(n)$ is an increasing function of $n \geq 0$ if $\beta < 0$ or $\alpha = 1/2$, or of $n \geq 2\beta$ if $\beta > \gamma$. Also, for $0 < \alpha < \frac{1}{2}$, $g_{\alpha\beta\gamma}(n)$ is an increasing function of

$$n \geq \max\left\{\frac{\gamma - \beta}{\alpha},\ 2\gamma\right\}. \tag{4.14}$$

These lower bounds for n suffice to reduce the proof of the Proposition for the exceptional algebras to a finite computer search.

Write $dim(\omega^f)$ for the Weyl dimension of the representation of X_ℓ with highest weight ω^f. The strategy is to use these simple results concerning when $g_{\alpha\beta\gamma}(n)$ is increasing with n, to find a level k_0 such that $\mathcal{D}_k(\lambda^k)$ is increasing (along each congruence class of k) for $k \geq k_0$. Running through all k-dependent candidates and their congruence classes, we obtain the following ranges for k_0: 2 to 2 for G_2; 7 to 7 for F_4; 10 to 10 for E_6; 14 to 14 for E_7; and 28 to 29 for E_8. Explicitly computing $\mathcal{D}_k(\lambda^k)$ for $k < k_0$, we find that in fact for each λ^k, $\mathcal{D}_k(\lambda^k)$ is monotonically increasing along each congruence class of k modulo $a_j^\vee$.

Now for each λ^k and each congruence class of k, let k_1 be the first level satisfying $dim(\omega^f) \leq \mathcal{D}_{k_1}(\lambda^{k_1})$. For k_1, we get the following ranges: 5 to 6 for G_2; 5 to 7 for F_4; 5 to 7 for E_6; 5 to 13 for E_7; and 7 to 31 for E_8.

We know from (4.6a) that $\mathcal{D}_k(\omega^f)$ is monotonically increasing; by the Weyl dimension formula it converges to $dim(\omega^f)$. Therefore we know $\mathcal{D}_k(\omega^f) < \mathcal{D}_k(\lambda^k)$ for all $k \geq k_1$. The remaining finitely many k can then be explicitly checked on a computer.

5. The Classical Algebras

In this section, we proceed to detail steps 2 and 3 of the proof of the Theorem, as outlined in section 2, for the four series of classical simple Lie algebras, with the exceptions of $B_{\ell,2}$ and $D_{\ell,2}$ which we consider in section 6. In each case, we first recall the relevant Lie algebraic data, and then explicitly give all automorphism invariants. The fusion products we need are computed using (3.6c).

5.1. The A–Series

All colabels $a_i^\vee$ are equal to 1, so that $h^\vee = \ell+1$ and a weight of $P_+(A_{\ell,k})$ satisfies $\lambda_0+\lambda_1+\ldots+\lambda_\ell = k$. The charge conjugation C acts as $C(\lambda) = (\lambda_0; \lambda_\ell, \lambda_{\ell-1}, \ldots, \lambda_1)$ and is trivial if $\ell = 1$.

The simple currents form a cyclic group of order $\ell+1$, generated by J with action $J(\lambda) = (\lambda_\ell; \lambda_0, \lambda_1, \ldots, \lambda_{\ell-1})$, corresponding to a rotation of the extended Dynkin diagram. Their charge and conformal weight are equal to $Q_{J^m}(\lambda) = \frac{m}{\ell+1}\sum_{j=1}^{\ell} j\lambda_j$ and $h_{J^m(0)} = km(\ell+1-m)/2(\ell+1)$.

Choose any positive integer m dividing $\ell+1$, such that $k(\ell+1)/m$ and m are coprime if m is odd, and such that $k(\ell+1)/2m$ is an integer coprime with m if m is even. In both cases, this means that we can find an integer v such that $vk(\ell+1)/2m \equiv 1 \bmod m$. To each such divisor m of $\ell+1$, one associates the automorphism invariant given by

$$\sigma_m(\lambda) = J^{-v(\ell+1)^2 Q_J(\lambda)/m}(\lambda), \tag{5A.1}$$

and first found in [13]. These and their conjugations $C \circ \sigma_m$ are the automorphisms appearing in Table 1; that they form the complete set is proved in [17]. The total number of different automorphism invariants equals 2^{c+p+t}, where

$$\begin{aligned}
c &= \begin{cases} -1 & \text{if } \ell = 1 \text{ and } k = 2, \\ 0 & \text{if } \ell = 1 \text{ and } k \neq 2, \text{ or } \ell \geq 2 \text{ and } k \leq 2, \\ 1 & \text{otherwise;} \end{cases} \\
p &= \text{number of distinct odd primes which divide } \ell+1 \text{ but not } k; \\
t &= \begin{cases} 0 & \text{if either } \ell \text{ is even, or } \ell \text{ is odd and } k \equiv 0 \bmod 4, \\ & \text{or } \ell \equiv 1 \bmod 4 \text{ and } k \text{ is odd}, \\ 1 & \text{otherwise.} \end{cases}
\end{aligned} \tag{5A.2}$$

All $A_{\ell,k}$ automorphism invariants have order 2 and commute.

5.2. The B–Series

A weight in $P_+(B_{\ell,k})$ satisfies $\lambda_0+\lambda_1+2\lambda_2+\ldots+2\lambda_{\ell-1}+\lambda_\ell=k$, and the dual Coxeter number of B_ℓ is $h^\vee=2\ell-1$. As $B_2\cong C_2$, we take $\ell\geq 3$.

The charge conjugation C is trivial, but there is a simple current of order 2, which exchanges the zero–th and first components, $J(\lambda)=(\lambda_1;\lambda_0,\lambda_2,\ldots,\lambda_\ell)$. It has $Q_J(\lambda)=\lambda_\ell/2$ and $h_{J(0)}=k/2$. When k is odd, there is the simple current automorphism invariant [3]

$$\sigma_J(\lambda)=J^{\lambda_\ell}(\lambda),\qquad \text{for } k \text{ odd}. \tag{5B.1}$$

As reported in Table 1, this is the only non–trivial invariant for $k\neq 2$, whereas for $k=2$, there are a number of exceptional invariants. As already apparent in Table 3, $k=2$ is very special, and we defer its full description to the next section. The case $k=1$ is straightforward (see [15]). We proceed here with the proof when $\ell\geq 3$ and $k\geq 3$.

From the corollary of section 4, we know that the action of any automorphism on the first fundamental weight is necessarily of the form $\sigma(\omega^1)=J^b(\omega^1)$. Suppose $b=1$. Then the norm condition yields

$$(\rho+J\omega^1)^2-(\rho+\omega^1)^2\equiv(k-2)n\equiv 0 \mod 2n. \tag{5B.2}$$

Therefore $\sigma(\omega^1)=J(\omega^1)$ requires k to be even.

The basic idea of the proof is the same as for the A_ℓ series in [17], but with the extra complication that not all fundamental representations are contained in fusion powers of ω^1. Thus we need a second weight, for which a convenient choice is the spinor ω^ℓ. The full proof (for $k\neq 2$) includes three steps:

(i) we first show that an automorphism which fixes ω^1 and ω^ℓ is necessarily trivial (this result also holds for $k=2$);

(ii) assuming that ω^1 is fixed, we find only four possibilities for $\sigma(\omega^\ell)$ consistent with the action of σ on the fusion product $\psi\times\omega^\ell$ (here $\psi=\omega^2$ is the adjoint representation); from this, we easily conclude that the only globally acceptable solutions are σ_1 (all k) and $\sigma=\sigma_J$ (k odd);

(iii) finally, we show that the assumptions k even and $\sigma(\omega^1)=J(\omega^1)$ are not compatible with σ being an automorphism of the fusion ring.

We first of all introduce the orthogonal basis $\{e_i\}$, convenient for computing fusion products. So we will set $\lambda=[x_1,x_2,\ldots,x_\ell]$, with the orthogonal components given in terms of the Dynkin components by

$$x_i=\lambda_i+\ldots+\lambda_{\ell-1}+\frac{\lambda_\ell}{2}, \tag{5B.3a}$$

$$x_\ell=\frac{\lambda_\ell}{2}. \tag{5B.3b}$$

In this basis, the metric is the identity $\lambda\cdot\lambda'=\sum_i x_ix'_i$, and the Weyl vector is $\rho=(1,\ldots,1)=[\ell-\frac{1}{2},\ell-\frac{3}{2},\ldots,\frac{3}{2},\frac{1}{2}]$.

(i)

We start off by proving that if ω^1 and ω^ℓ are both fixed by σ, then all weights are fixed, so that $\sigma=\sigma_1$. The weights of the defining representation ω^1 are $\{0,\ \pm e_i\}_{1\leq i\leq\ell}$, so that

$$\omega^1\times\omega^1=0+\omega^2+(2\omega^1), \tag{5B.4a}$$

$$\omega^1\times\omega^i=\omega^{i-1}+\omega^{i+1}+(\omega^1+\omega^i),\qquad \text{for } 2\leq i\leq\ell-2. \tag{5B.4b}$$

The norms of the weights appearing in (5B.4) read $(\rho+\omega^i)^2=\rho^2+i(2\ell+1-i)$ for $1\leq i\leq\ell-1$, $(\rho+2\omega^1)^2=\rho^2+4\ell+2$ and $(\rho+\omega^1+\omega^i)^2=\rho^2+i(2\ell+1-i)+2\ell+2$, also for $1\leq i\leq\ell-1$. Assuming $\sigma(\omega^1)=\omega^1$, we obtain that σ must permute the weights on the r.h.s. of (5B.4a). But a non–trivial

permutation is forbidden by the values of their norms, so that $\sigma(\omega^2)=\omega^2$. The same argument applies to (5B.4b) with $i=2$, showing that ω^3 must be fixed by σ, and by induction, all weights ω^i, $i<\ell$, must be fixed. If ω^ℓ is assumed to be fixed as well, then Lemma 1 implies that the whole of the alcôve is fixed, and that $\sigma=\sigma_1$.

(ii)
Here we assume that ω^1 is fixed by σ, and show that the only automorphisms with this property are $\sigma=\sigma_1$ and, for k odd, $\sigma=\sigma_J$.

The fusion (5B.4a) shows that the adjoint $\psi=\omega^2=[1,1,0,\ldots,0]$ must be fixed by σ. We first compute the fusion of ψ with the spinor $\omega^\ell=[\frac{1}{2},\ldots,\frac{1}{2}]$, then compare it with that of ψ with $\sigma(\omega^\ell)$ and require they be compatible.

The weights of the spinor representation ω^ℓ are $P(\omega^\ell)=\{[\pm\frac{1}{2},\ldots,\pm\frac{1}{2}]\}$ (with uncorrelated signs), so that the weights appearing in $\psi\times\omega^\ell$ have the following form: *(a)* in the first two positions, there will be $\frac{1}{2}$'s and $\frac{3}{2}$'s, but a $\frac{1}{2}$ followed by a $\frac{3}{2}$ puts the weight in a wall of the alcôve (meaning it would be fixed by a Weyl reflection and so does not contribute), and *(b)* in the last $\ell-2$ positions, there will be $\frac{1}{2}$'s and $-\frac{1}{2}$'s, but a $-\frac{1}{2}$ followed by a $\frac{1}{2}$ or an ending $-\frac{1}{2}$ also puts the weight in a wall (recall that we are using weights non–shifted by ρ). Thus

$$\psi\times\omega^\ell=\omega^\ell+[\frac{3}{2},\frac{1}{2},\ldots,\frac{1}{2}]+[\frac{3}{2},\frac{3}{2},\frac{1}{2},\ldots,\frac{1}{2}]. \tag{5B.5}$$

Set $\lambda=\sigma(\omega^\ell)$. The weight diagram of the adjoint is the set of roots of B_ℓ so that

$$P(\psi)+\lambda=\{\lambda\pm e_i,\ \lambda\pm(e_i-e_j),\ \lambda\pm(e_i+e_j),\ \lambda\}_{1\le i<j\le\ell}. \tag{5B.6}$$

From (5B.5) and (3.8d), we require that $N_{\psi,\lambda}^{\lambda}=1$. But $\mathrm{mult}_\psi(0)=\ell$, and this implies from (3.6c) that there should be $\ell-1$ non–zero roots α such that the weights $\lambda-\alpha$ get out of the alcôve and brought back onto λ by an odd Weyl transformation. Looking at all non–zero roots, we find that those which can take $\lambda-\alpha$ out of the alcôve and off the walls are

1. the $\ell+1$ affine simple roots α_i ($\alpha_0=-\psi$) iff $\lambda\cdot\alpha_i=0$ for $i\ge1$ (*i.e.* the i–th Dynkin label equal to zero), and $\lambda\cdot\psi=k$ for $i=0$. One easily checks that $w_i(\rho+\lambda-\alpha_i)=\rho+\lambda$ with w_i the Weyl reflector through the i–th hyperplane. [For $i=0$, the reflection is given by $w_0(\lambda)=\lambda+(n-\lambda\cdot\psi)\psi$.]
2. the roots $\alpha=\pm e_i+e_\ell$ for $1\le i\le\ell-1$ iff $\lambda\cdot\alpha_\ell=x_\ell=0$. In this case, we have $w_\ell(\rho+\lambda-\alpha)=\rho+\lambda\mp e_i$, so that the weights $\lambda\pm e_i-e_\ell$ and $\lambda\pm e_i$ all cancel out.

Since the condition $N_{\psi,\lambda}^{\lambda}=1$ requires that $(\ell-1)$ λ's cancel against some $\lambda-\alpha_i$ for some choice of $(\ell-1)$ affine simple roots α_i, we find that λ must have either $\ell-1$ zero Dynkin labels and satisfy $\lambda\cdot\psi=x_1+x_2<k$, or else $\ell-2$ zero Dynkin labels and satisfy $\lambda\cdot\psi=x_1+x_2=k$. In addition, the fusion $\psi\times\lambda$ must contain exactly three weights. From (5B.6), the result is that these two conditions, $N_{\psi,\lambda}^{\lambda}=1$ and $\sum_\mu N_{\psi,\lambda}^{\mu}=3$, force λ to be one of the following four weights

$$\lambda=\sigma(\omega^\ell)\in\big\{\omega^1,\ J(\omega^1),\ \omega^\ell,\ J(\omega^\ell)\big\}. \tag{5B.7}$$

The first weight in (5B.7) must be discarded since it was assumed to be fixed by σ. If $\lambda=J(\omega^1)$, then using $N_{\lambda,J(\mu)}^{J(\nu)}=N_{\lambda,\mu}^{\nu}$, a straightforward consequence of $S_{\lambda,J(\lambda')}=\mathrm{e}^{2i\pi x_\ell}\,S_{\lambda,\lambda'}$ and the Verlinde formula, we obtain from (5B.4a)

$$\omega^1\times J(\omega^1)=[k-2,0,\ldots,0]+[k-1,1,0,\ldots,0]+[k,0,\ldots,0], \tag{5B.8}$$

which must be the transform by σ of

$$\omega^1\times\omega^\ell=\omega^\ell+[\frac{3}{2},\frac{1}{2},\ldots,\frac{1}{2}], \tag{5B.9}$$

clearly impossible.

The solution $\lambda = \sigma(\omega^\ell) = \omega^\ell$ leads to $\sigma = \sigma_1$ by step *(i)*, since σ then fixes both ω^1 and ω^ℓ. The remaining possibilitiy $\lambda = J(\omega^\ell)$ requires k odd for norm reasons, and leads to the simple current automorphism $\sigma = \sigma_J$ of (5B.1). Indeed $\sigma_J^{-1} \circ \sigma$, fixing ω^1 and ω^ℓ, must be trivial, implying $\sigma = \sigma_J$.

(iii)
We now assume that k is even and that $\sigma(\omega^1) = J(\omega^1)$. With the results of step *(ii)*, it is easy to show that these assumptions lead to a contradiction: for $k \geq 4$ even, we will find that there is no automorphism such that $\sigma(\omega^1) = J(\omega^1)$.

The identity $N^\nu_{J(\lambda),J(\mu)} = N^\nu_{\lambda,\mu}$ implies from (5B.4a)

$$\omega^1 \times \omega^1 = J(\omega^1) \times J(\omega^1) = 0 + \omega^2 + (2\omega^1). \tag{5B.10}$$

As before, we obtain that the adjoint ψ must be fixed, since σ must preserve the r.h.s. of (5B.10). The argument we used in the second part *(ii)* then shows that $\lambda = \sigma(\omega^\ell)$ must be one of the four weights in (5B.7).

Take first $\lambda = \omega^1$. This again implies that the fusion product (5B.8) must be the σ–transform of that in (5B.9), which is impossible. The second weight $\lambda = J(\omega^1)$ in (5B.7) must also be discarded since $J(\omega^1) = \sigma(\omega^1)$ is already the image of ω^1.

If $\lambda = \omega^\ell$, we obtain from (5B.9), using once more $N^{J(\nu)}_{J(\lambda),\mu} = N^\nu_{\lambda,\mu}$,

$$J(\omega^1) \times \omega^\ell = [k - \frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2}] + [k - \frac{3}{2}, \frac{1}{2}, \ldots, \frac{1}{2}]. \tag{5B.11}$$

Again (5B.11) must be the image under σ of (5B.9). This requires that $\omega^\ell = [\frac{1}{2}, \ldots, \frac{1}{2}]$ be in the r.h.s. of (5B.11), implying $k = 1$ or $k = 2$, contrary to the assumption $k \geq 4$.

Finally, $\lambda = J(\omega^\ell)$ requires k odd.

Therefore, all four possibilities in (5B.7) lead to a contradiction, and the proof of the Theorem is complete for the $B_{\ell,k}$ algebras, $k \neq 2$. ∎

5.3. *The C–Series*

A weight of $P_+(C_{\ell,k})$ satisfies $\lambda_0 + \lambda_1 + \ldots + \lambda_\ell = k$ and the dual Coxeter number is equal to $h^\vee = \ell + 1$. Here too, the charge conjugation C is trivial, and there is one simple current J, of order 2, defined by $J(\lambda) = (\lambda_\ell; \lambda_{\ell-1}, \ldots, \lambda_1, \lambda_0)$. It has $h_{J(0)} = k\ell/4$ and $Q_J(\lambda) = \sum_{j=1}^{\ell} j\lambda_j/2$.

When $k\ell \equiv 2 \bmod 4$, there is a simple current automorphism invariant given by [3]

$$\sigma_J(\lambda) = J^{2Q_J(\lambda)}(\lambda), \qquad \text{if } k\ell \equiv 2 \bmod 4. \tag{5C.1}$$

The diagonal invariant σ_1 and σ_J are the only automorphism invariants (note that for $\ell = 2$ and $k = 1$, $\sigma_J = \sigma_1$).

Let σ be any automorphism invariant of $C_{\ell,k}$. From the corollary of section 4, we have that, for any k, $\sigma(\omega^1) = J^b(\omega^1)$ for some $b = 0, 1$. Suppose $b = 1$. Then the norm condition yields

$$(\rho + \omega^1)^2 \equiv (\rho + J\omega^1)^2 \bmod 2n \quad \Rightarrow \quad \frac{1}{2}(k\ell - 2)n \equiv 0 \bmod 2n. \tag{5C.2}$$

Therefore $\sigma(\omega^1) = J(\omega^1)$ requires $k\ell \equiv 2 \bmod 4$. But precisely for those values of k and ℓ, there exists the automorphism invariant σ_J, whose action on ω^1 is also $\sigma(\omega^1) = J(\omega^1)$. Thus replacing σ by $\sigma_J^{-1} \circ \sigma$, we may assume for all k that $\sigma(\omega^1) = \omega^1$, and show that the only such automorphism is trivial. This will complete the proof for the C_ℓ series.

The fusion of ω^1 with the other fundamentals reads

$$\omega^1 \times \omega^1 = 0 + \omega^2 + (2\omega^1), \tag{5C.3a}$$
$$\omega^1 \times \omega^i = \omega^{i-1} + \omega^{i+1} + (\omega^1 + \omega^i), \quad \text{for } 2 \le i \le \ell - 1. \tag{5C.3b}$$

The norms of the weights in the r.h.s. of (5C.3a–b) are equal to $(\rho+\omega^i)^2 = \rho^2 + i(\ell+1-\frac{i}{2})$, $(\rho+2\omega^1)^2 = \rho^2 + 2\ell + 2$, and $(\rho+\omega^1+\omega^i)^2 = \rho^2 + i(\ell+1-\frac{i}{2}) + \ell + \frac{3}{2}$. Assuming that ω^1 is fixed by σ, the norm argument shows from (5C.3a) that ω^2 must be fixed, and then from (5C.3b), that all ω^i are fixed as well. In turn, Lemma 1 implies that all weights of $P_+(C_{\ell,k})$ must be fixed, and $\sigma = \sigma_1$. ■

5.4. The D–Series

A weight of $P_+(D_{\ell,k})$ satisfies $\lambda_0+\lambda_1+2\lambda_2+\ldots+2\lambda_{\ell-2}+\lambda_{\ell-1}+\lambda_\ell = k$, and the height is $n = k+2\ell-2$. Since $D_3 \cong A_3$, we will assume $\ell \ge 4$.

For any ℓ, there is the outer automorphism

$$C_1(\lambda) = (\lambda_0; \lambda_1, \ldots, \lambda_{\ell-2}, \lambda_\ell, \lambda_{\ell-1}). \tag{5D.1a}$$

For ℓ odd, $C = C_1$ is the charge conjugation, while for ℓ even, the charge conjugation is trivial. Moreover when $\ell = 4$, there are four new outer automorphisms given by

$$C_2(\lambda) = (\lambda_0; \lambda_4, \lambda_2, \lambda_3, \lambda_1), \quad C_3 = C_1C_2, \quad C_4 = C_2C_1, \quad C_5 = C_1C_2C_1. \tag{5D.1b}$$

Together with $C_0 = \sigma_1$, these six C_i correspond to the different permutations of the Dynkin labels $\lambda_1, \lambda_3, \lambda_4$.

There are three non–trivial simple currents, J_v, J_s and $J_c = J_v \circ J_s$. Explicitly, we have

$$J_v\lambda = (\lambda_1; \lambda_0, \lambda_2, \ldots, \lambda_{\ell-2}, \lambda_\ell, \lambda_{\ell-1}), \tag{5D.2a}$$
$$Q_v(\lambda) = (\lambda_{\ell-1}+\lambda_\ell)/2, \quad h_{J_v(0)} = k/2. \tag{5D.2b}$$

The expressions for J_s and J_c depend on the parity of ℓ and are given by

$$J_s(\lambda) = \begin{cases} (\lambda_\ell; \lambda_{\ell-1}, \lambda_{\ell-2}, \ldots, \lambda_1, \lambda_0) & \text{if } \ell \text{ is even,} \\ (\lambda_{\ell-1}; \lambda_\ell, \lambda_{\ell-2}, \ldots, \lambda_1, \lambda_0) & \text{if } \ell \text{ is odd,} \end{cases} \tag{5D.2c}$$
$$Q_s(\lambda) = \sum_{j=1}^{\ell-2} j\lambda_j/2 - \frac{\ell-2}{4}\lambda_{\ell-1} - \frac{\ell}{4}\lambda_\ell, \quad h_{J_s(0)} = k\ell/8, \tag{5D.2d}$$

and

$$J_c(\lambda) = \begin{cases} (\lambda_{\ell-1}; \lambda_\ell, \lambda_{\ell-2}, \ldots, \lambda_2, \lambda_0, \lambda_1) & \text{if } \ell \text{ is even,} \\ (\lambda_\ell; \lambda_{\ell-1}, \lambda_{\ell-2}, \ldots, \lambda_2, \lambda_0, \lambda_1) & \text{if } \ell \text{ is odd,} \end{cases} \tag{5D.2e}$$
$$Q_c(\lambda) = \sum_{j=1}^{\ell-2} j\lambda_j/2 - \frac{\ell}{4}\lambda_{\ell-1} - \frac{\ell-2}{4}\lambda_\ell, \quad h_{J_c(0)} = k\ell/8. \tag{5D.2f}$$

All three simple currents have order 2, except J_s and J_c which have order 4 if ℓ is odd. We denote by N_s the order of J_s (equal to the order of J_c).

Corresponding to these simple currents, one defines the following simple current automorphism invariants

$$\sigma_v(\lambda) = J_v^{\lambda_{\ell-1}+\lambda_\ell}(\lambda), \qquad \text{if } k \text{ is odd,} \tag{5D.3a}$$
$$\sigma_s(\lambda) = J_s^{N_s^2 k\ell\, Q_s(\lambda)\,/8}(\lambda), \qquad \text{if } N_s k\ell \equiv 8 \bmod 16, \tag{5D.3b}$$
$$\sigma_c(\lambda) = J_c^{N_s^2 k\ell\, Q_c(\lambda)\,/8}(\lambda), \qquad \text{if } N_s k\ell \equiv 8 \bmod 16, \tag{5D.3c}$$

with $\sigma_s = \sigma_c$ if ℓ is odd. The automorphism invariant σ_v was found in [3] as well as σ_s and σ_c for ℓ even, while σ_s for ℓ odd was discovered in [1].

The last simple current automorphism invariant for $D_{\ell,k}$, when k and ℓ are both even and $k\ell \equiv 0 \bmod 8$, was found in [30]. It is the integer spin simple current automorphism we mentioned in section 3, and explicitly reads

$$\sigma_{vsc}(\lambda) = \begin{cases} \lambda & \text{if } Q_v(\lambda) \equiv 0,\ Q_s(\lambda) \equiv 0 \bmod 1, \\ J_v(\lambda) & \text{if } Q_v(\lambda) \equiv 0,\ Q_s(\lambda) \equiv \frac{1}{2} \bmod 1, \\ J_s(\lambda) & \text{if } Q_v(\lambda) \equiv \frac{1}{2},\ Q_s(\lambda) \equiv 0 \bmod 1, \\ J_c(\lambda) & \text{if } Q_v(\lambda) \equiv \frac{1}{2},\ Q_s(\lambda) \equiv \frac{1}{2} \bmod 1. \end{cases} \tag{5D.3d}$$

Obviously any product of these with each other (when the values of k and $k\ell$ allow it) and with the C_i will define other automorphism invariants. Together, they generate all of them, for $k \neq 2$.

When $k \equiv \ell \equiv 2 \bmod 4$, σ_s and σ_c generate an Abelian subgroup of order 4, containing the elements $\sigma_1, \sigma_s, \sigma_c$, and $\sigma_s \circ \sigma_c = \sigma_c \circ \sigma_s$. In this case the automorphism invariants are just $C_1^a \sigma_s^b \sigma_c^c$, where $a, b, c = 0, 1$.

When $\ell \equiv 4 \bmod 8$ and k is odd, the subgroup generated by σ_s and σ_c is of order 6, and consists of the elements $\sigma_1, \sigma_s, \sigma_c, \sigma_s \circ \sigma_c, \sigma_c \circ \sigma_s$, and $\sigma_s \circ \sigma_c \circ \sigma_s = \sigma_c \circ \sigma_s \circ \sigma_c = \sigma_v$. Any automorphism invariant in this case will look like $C_j\sigma$, where $\sigma \in \langle \sigma_s, \sigma_c \rangle$ and C_j is one of the 2 ($\ell \neq 4$) or 6 ($\ell = 4$) conjugations.

In general we have $C_1\sigma_v = \sigma_v C_1$, $C_j \sigma_{vsc} = \sigma_{vsc} C_j$ and $C_1 \sigma_s = \sigma_c C_1$. Also, $\sigma_v^2 = \sigma_s^2 = \sigma_c^2 = \sigma_{vsc}^2 = \sigma_1$.

When $k = 2$, there are in addition a number of exceptional invariants, detailed in the next section, and first found in [12]. The proof for $k = 1$ was done in [15].

We now proceed to show, for $k \geq 3$, that this list of automorphism invariants, also summarized in Table 1, is exhaustive. As usual, we first use the results of section 4 to restrict the possible values of $\sigma(\omega^1)$.

Let σ be any automorphism invariant of $D_{\ell,k}$. From the corollary of section 4, we have that, for any $k \neq 2$, $\sigma(\omega^1) = C_j J_s^a J_v^b(\omega^1)$, where C_j is some conjugation, and $a, b = 0, 1$. By replacing σ with $C_j \circ \sigma$, we may drop C_j.

Consider first the possibility $a = 1, b = 0$. Then

$$(\rho + \omega^1)^2 \equiv (\rho + J_s\omega^1)^2 \mod 2n \quad \Rightarrow \quad (\frac{k\ell}{4} - 1)\, n \equiv 0 \mod 2n. \tag{5D.4}$$

Therefore $\sigma(\omega^1) = J_s(\omega^1)$ requires $k\ell \equiv 4 \bmod 8$. When ℓ is even, there exists an automorphism invariant σ_s for these k, ℓ with the property that $\sigma_s(\omega^1) = J_s(\omega^1)$; in this case, replacing σ with $\sigma_s \circ \sigma$, we may assume $\sigma(\omega^1) = \omega^1$. On the other hand, when ℓ is odd, the charge conjugation $C_1 = S^2$ must commute with any automorphism invariant: *i.e.* $C_1 \circ \sigma \circ C_1 = \sigma$. But this would be violated if $\sigma(\omega^1) = J_s(\omega^1)$, since $J_s(\omega^1) = J_c(\omega^1)$ only happens when $k = 2$. Thus the case $a = 1, b = 0$ cannot apply when ℓ is odd. The identical argument applies to $a = b = 1$.

Finally, consider $a = 0, b = 1$. In this case

$$(\rho + \omega^1)^2 \equiv (\rho + J_v\omega^1)^2 \mod 2n \quad \Rightarrow \quad (k-2)n \equiv 0 \mod 2n. \tag{5D.5}$$

Therefore $\sigma(\omega^1) = J_v(\omega^1)$ requires k even. If in addition $k\ell \equiv 2 \bmod 4$, then replace σ with $\sigma_s \circ \sigma$. If instead ℓ is also even, and $k\ell \equiv 0 \bmod 8$, then replace σ with $\sigma_{vsc} \circ \sigma$. Finally, if ℓ is even and $k\ell \equiv 4 \bmod 8$, then replace σ with $\sigma_s \circ \sigma_c \circ \sigma$. So in all cases, after composing it with adequate automorphisms, one may assume $\sigma(\omega^1) = \omega^1$ *except* if both $k \equiv 0 \bmod 4$ and ℓ is odd, in which case $\sigma(\omega^1) = J_v(\omega^1)$ remains a possibility.

The idea of the proof is exactly the same as in the B_ℓ series, and is only slightly more complicated due to the larger number of outer automorphisms. More precisely, for $k \neq 2$, we will go through the proof of the following three points:

(i) an automorphism which fixes ω^1 and the spinor ω^ℓ must be trivial (true even for $k = 2$);

(ii) assuming that ω^1 alone is fixed, there are now twelve possibilities for $\sigma(\omega^\ell)$ consistent with the action of σ on the fusion product $\psi \times \omega^\ell$; apart from the trivial solution $\sigma = \sigma_1$, this will imply that the only globally acceptable solutions are C_1 (all k) and σ_v, $C_1\sigma_v$ (for k odd);

(iii) finally, the assumptions $k \equiv 0 \bmod 4$, ℓ odd and $\sigma(\omega^1) = J_v(\omega^1)$ are not compatible with σ being an automorphism of the fusion ring.

Again we first introduce an orthogonal basis $\{e_i\}$ in the weight space, and write $\lambda = [x_1, x_2, \ldots, x_\ell]$ with the new components given in terms of the Dynkin labels as

$$x_i = \lambda_i + \ldots + \lambda_{\ell-2} + \frac{1}{2}(\lambda_{\ell-1} + \lambda_\ell), \tag{5D.6a}$$

$$x_{\ell-1} = \frac{1}{2}(\lambda_{\ell-1} + \lambda_\ell), \tag{5D.6b}$$

$$x_\ell = \frac{1}{2}(-\lambda_{\ell-1} + \lambda_\ell). \tag{5D.6c}$$

In that basis, the metric is the identity $\lambda \cdot \lambda' = \sum_i x_i x_i'$, and the Weyl vector reads $\rho = (1, \ldots, 1) = [\ell-1, \ell-2, \ldots, 1, 0]$.

(i)

Let us show that if ω^1 and ω^ℓ are both assumed to be fixed by a σ, then all weights are fixed as well and $\sigma = \sigma_1$. The weight diagram of the defining representation is the set $P(\omega^1) = \{\pm e_i\}_{1 \le i \le \ell}$, and we obtain the following fusions

$$\omega^1 \times \omega^1 = 0 + \omega^2 + (2\omega^1), \tag{5D.7a}$$

$$\omega^1 \times \omega^i = \omega^{i-1} + \omega^{i+1} + (\omega^1 + \omega^i), \quad \text{for } 2 \le i \le \ell - 3, \tag{5D.7b}$$

$$\omega^1 \times \omega^\ell = \omega^{\ell-1} + (\omega^1 + \omega^\ell). \tag{5D.7c}$$

The usual norm argument applies once more. The first fusion (5D.7a) implies that ω^2 is fixed by σ if ω^1 is fixed. Then (5D.7b) shows that all fundamental weights ω^i, for $3 \le i \le \ell-2$, are fixed. Finally, assuming ω^ℓ fixed, the last fusion forces $\omega^{\ell-1}$ to be fixed as well. From Lemma 1, the conclusion follows that all weights in $P_+(D_{\ell,k})$ are invariant, or $\sigma = \sigma_1$.

(ii)

We assume here that $\sigma(\omega^1) = \omega^1$ and classify all automorphisms with that property, for $k \ge 3$.

The fusion (5D.7a) shows that the adjoint $\psi = \omega^2 = [1, 1, 0, \ldots, 0]$ must be fixed by any σ which leaves ω^1 invariant. We will compute the fusions $\psi \times \omega^\ell$ and $\psi \times \sigma(\omega^\ell)$ and require their compatibility, thereby restricting $\sigma(\omega^\ell)$.

The weight diagram of the spinor ω^ℓ is $P(\omega^\ell) = \{[\pm\frac{1}{2}, \ldots, \pm\frac{1}{2}]\}$ where the number of $-$ signs is even. Arguing as in the B_ℓ case, we find

$$\psi \times \omega^\ell = \omega^\ell + [\frac{3}{2}, \frac{1}{2}, \ldots, \frac{1}{2}, -\frac{1}{2}] + [\frac{3}{2}, \frac{3}{2}, \frac{1}{2}, \ldots, \frac{1}{2}]. \tag{5D.8}$$

Denote $\lambda = \sigma(\omega^\ell)$. The weight diagram of the adjoint is the set of roots of D_ℓ so that

$$P(\psi) + \lambda = \{\lambda \pm (e_i - e_j),\ \lambda \pm (e_i + e_j),\ \lambda\}_{1 \le i < j \le \ell}. \tag{5D.9}$$

Again, $\mathrm{mult}_\psi(0) = \ell$. As in B_ℓ, this implies that there must be $\ell - 1$ non–zero roots α such that $\lambda - \alpha$ gets out of the alcôve and is mapped back on λ by an odd Weyl transformation. In this case, we find that the only non–zero roots which can take $\lambda - \alpha$ out of the alcôve are the $\ell + 1$ affine simple roots α_i, with $\alpha_0 = -\psi$. Moreover $\lambda - \alpha_i$ is out of the alcôve if and only if $\lambda \cdot \alpha_i = 0$ for $i \ge 1$, and $\lambda \cdot \psi = k$ for $i = 0$. One also checks that $w_i(\rho + \lambda - \alpha_i) = \rho + \lambda$ with w_i the Weyl reflector through the i–th hyperplane. Therefore, $N_{\psi,\lambda}^{\lambda} = 1$ if

and only if either $\ell-1$ Dynkin labels are zero and $\lambda\cdot\psi = x_1+x_2 < k$, or else $\ell-2$ Dynkin labels are zero and $\lambda\cdot\psi = x_1+x_2 = k$. The other condition we obtain by comparing (5D.8) and (5D.9) is that the fusion $\psi\times\lambda$ must contain exactly three weights. Altogether the two conditions $N_{\psi,\lambda}^{\lambda}=1$ and $\sum_\mu N_{\psi,\lambda}^{\mu}=3$ force $\lambda=\sigma(\omega^\ell)$ to be one of the following twelve weights (given in the Dynkin basis)

$$\lambda = \omega^\ell,\ \omega^{\ell-1},\ (k-1,0,\ldots,0,1,0),\ (k-1,0,\ldots,0,1), \tag{5D.10a}$$
$$\lambda = \omega^1,\ (k-1,0,\ldots,0), \tag{5D.10b}$$
$$\begin{aligned}\lambda &= (0,\ldots,0,1,k-1),\ (0,\ldots,0,k-1),\ (1,0,\ldots,0,k-1,0),\\ &\quad\text{and } (0,\ldots,0,k-1,1),\ (0,\ldots,0,k-1,0),\ (1,0,\ldots,0,k-1).\end{aligned} \tag{5D.10c}$$

It remains to examine these 12 possibilities case by case.

The four weights in (5D.10a) correspond to $\lambda=\sigma(\omega^\ell)$ with σ given respectively by $\sigma=1$, C_1, σ_v and $C_1\sigma_v$ (the last two requiring k odd for norm reasons). These four automorphisms all leave ω^1 fixed, so that composing them with σ leaves us with an automorphism which fixes both ω^1 and ω^ℓ, hence trivial by step *(i)*. This shows that $\sigma=\sigma_1$, C_1, σ_v and $C_1\sigma_v$ everywhere.

The possibility $\lambda=\omega^1$ must be discarded since ω^1 was assumed to be fixed. The second one, $\lambda = (k-1,0,\ldots,0) = J_v(\omega^1)$, must also be excluded. Indeed the identity $N_{\lambda,J_v(\mu)}^{J_v(\nu)} = N_{\lambda,\mu}^{\nu}$ applied to $\omega^1\times\omega^1$ yields

$$\omega^1\times J_v(\omega^1) = J_v(0)+J_v(\omega^2)+J_v(2\omega^1). \tag{5D.11}$$

This must be the image under σ of the product $\omega^1\times\omega^\ell$ given in (5D.7c), and which contains only two fields in its r.h.s., leading to a contradiction.

As to the six weights in (5D.10c), it is enough to consider the first three, $\lambda^1 := (0,\ldots,0,1,\ k-1)$, $\lambda^2 := (0,\ldots,0,k-1)$ and $\lambda^3 := (1,0,\ldots,0,k-1,0)$, since the last three are their conjugates by C_1. Let us compare the fusions of ω^1 with ω^ℓ and with λ^i :

$$\omega^1\times\omega^\ell = (\omega^\ell+e_1)+(\omega^\ell-e_\ell), \tag{5D.12a}$$
$$\omega^1\times\lambda^1 = (\lambda^1+e_\ell)+(\lambda^1-e_{\ell-1})+(\lambda^1-e_\ell), \tag{5D.12b}$$
$$\omega^1\times\lambda^2 = (\lambda^2+e_1)+(\lambda^2-e_\ell), \tag{5D.12c}$$
$$\omega^1\times\lambda^3 = (\lambda^3-e_1)+(\lambda^3+e_\ell). \tag{5D.12d}$$

Assuming $(\rho+\omega^\ell)^2 \equiv (\rho+\lambda^i)^2 \bmod 2n$ for consistency, one obtains the norms

$$(\rho+\omega^\ell-e_\ell)^2 = (\rho+\omega^\ell)^2, \tag{5D.13a}$$
$$(\rho+\lambda^2+e_1)^2 \equiv (\rho+\lambda^3-e_1)^2 \equiv (\rho+\omega^\ell)^2+n \bmod 2n, \tag{5D.13b}$$
$$(\rho+\lambda^2-e_\ell)^2 \equiv (\rho+\lambda^3+e_\ell)^2 \equiv (\rho+\omega^\ell)^2+2-k \bmod 2n. \tag{5D.13c}$$

From (5D.12a) and (5D.12b), we see $\sigma(\omega^\ell)\neq\lambda^1$. Comparing (5D.12a) with (5D.12c), we obtain either $\sigma(\omega^\ell-e_\ell)=\lambda^2+e_1$ or λ^2-e_ℓ. But the former is ruled out by the norm condition (5D.13b), and the latter leads by (5D.13c) to $k=2$, contrary to the assumption $k\geq 3$. Thus λ^2 must also be excluded. The weight λ^3 is similarly ruled out, because the norm condition either leads to a contradiction, or else forces $k=2$.

(iii)
We finish the proof by showing that there is no automorphism satisfying $\sigma(\omega^1)=J_v(\omega^1)$ if both $k\equiv 0 \bmod 4$ and ℓ odd.

As in step *(ii)*, the fusion

$$\omega^1\times\omega^1 = J_v(\omega^1)\times J_v(\omega^1) = 0+\psi+(2\omega^1) \tag{5D.14}$$

shows that ψ must be fixed, consequently that $\lambda=\sigma(\omega^\ell)$ must be one of the twelve weights in (5D.10) (see the argument in step *(ii)*).

Assume first $\lambda = \omega^\ell$. Using $N^{J_v(\nu)}_{J_v(\lambda),\mu} = N^{\nu}_{\lambda,\mu}$ leads to

$$\omega^1 \times \omega^\ell = (\omega^\ell + e_1) + (\omega^\ell - e_\ell), \tag{5D.15a}$$
$$J_v(\omega^1) \times \omega^\ell = (\omega^\ell + (k-1)e_1) + (\omega^{\ell-1} + (k-2)e_1). \tag{5D.15b}$$

Trying to match the r.h.s. of (5D.15a) and (5D.15b), the norm forces either k odd or $k=2$. Thus $\lambda = \omega^\ell$ is impossible, as is its conjugate $\lambda = \omega^{\ell-1}$.

The next two possibilities, $\lambda = (k-1, 0, \ldots, 0, 1, 0)$ and its conjugate, correspond to $\lambda = J_v(\omega^\ell)$ and $\lambda = C_1 J_v(\omega^\ell)$, which require k odd.

The case $\lambda = (k-1, 0, \ldots, 0) = J_v(\omega^1)$ is clearly impossible since it is already the image of ω^1. As to $\lambda = \omega^1$, it is forbidden for the same reason as in step *(ii)*, namely because $\sigma(\omega^1 \times \omega^\ell) = \omega^1 \times J_v(\omega^1)$ and that the two fusions do not contain the same number of fields, see (5D.7c) and (5D.11).

The first and fourth weights of (5D.10c) are ruled out as in step *(ii)*. From (5D.12b), we obtain ($\lambda^1 = (0, \ldots, 0, 1, k-1)$)

$$J_v(\omega^1) \times \lambda^1 = J_v(\lambda^1 + e_\ell) + J_v(\lambda^1 - e_{\ell-1}) + J_v(\lambda^1 - e_\ell). \tag{5D.16}$$

Since there are only two weights on the r.h.s. of (5D.12a), (5D.16) implies $\sigma(\omega^\ell) \neq \lambda^1$. Similarly, $\sigma(\omega^\ell) \neq (0, \ldots, 0, k-1, 1)$.

There now remain four weights in (5D.10c), namely $\lambda^2 = (0, \ldots, 0, k-1)$, $\lambda^3 = (1, 0, \ldots, 0, k-1, 0)$, and their C_1–conjugates. But the norm condition implies $\frac{k\ell}{2} \equiv \ell \bmod 4$ if $\sigma(\omega^\ell) = \lambda^2$, and $\frac{k\ell}{2} \equiv \ell + 2 \bmod 4$ if $\sigma(\omega^\ell) = \lambda^3$, and these congruences are not consistent with the assumptions ℓ odd and $k \equiv 0 \bmod 4$ that we made. This finishes the proof of step *(iii)*, and that of the Theorem for the $D_{\ell,k}$ algebras, $k \neq 2$. ∎

6. The Orthogonal Algebras, Level 2

As already clear from Table 3, something special happens for the orthogonal algebras when the level k is equal to 2: a large number of fields have equal quantum dimensions. This has the immediate consequence that the technique we used in the previous section is no longer available. More importantly however it hints at the fact that the current algebras $B_{\ell,2}$ and $D_{\ell,2}$ have a much richer spectrum of modular invariants than at the other levels. Indeed, exceptional automorphism invariants have been recently discovered in [12] for most orthogonal algebras, level 2. It is the purpose of this section to show that the list of automorphisms anticipated in [12] form the complete set, and to give a detailed description of them.

6.1. The B–Series, Level 2

The alcôve $P_+(B_{\ell,2})$ contains $\ell + 4$ weights: the identity, the ℓ fundamental weights ω^i, and the three combinations $2\omega^1$, $\omega^1 + \omega^\ell$ and $2\omega^\ell$. For what follows, it is convenient to rename ℓ of these weights as

$$\nu^i := \omega^i, \quad \text{for } 1 \le i \le \ell - 1 \qquad \text{and} \qquad \nu^\ell := 2\omega^\ell\,. \tag{6.1}$$

At level 2, the height for $B_{\ell,2}$ is equal to $n = 2\ell + 1$.

For any number x, we define $[x]_n$ to be the unique number $0 \le [x]_n \le \frac{n}{2}$ satisfying $x \equiv \pm [x]_n \bmod n$ for some choice of sign. Then for each integer m satisfying $m^2 \equiv 1 \mod n$, we define the following permutation of $P_+(B_{\ell,2})$

$$\sigma_m \;:\; \begin{cases} \sigma_m(\nu^i) = \nu^{[mi]_n} & \text{for all } 1 \le i \le \ell, \\ \sigma_m(\lambda) = \lambda & \text{if } \lambda \in \{0, 2\omega^1, \omega^1 + \omega^\ell, \omega^\ell\}. \end{cases} \tag{6.2}$$

We leave the proof that the σ_m actually define automorphism invariants for section 6.3, where we interpret them as generalized Galois automorphisms. Note that, since $\sigma_m(\nu^1) = \nu^{[m]_n}$, we obtain that

$\sigma_m = \sigma_{m'}$ if and only if $[m]_n = [m']_n$, or equivalently $m \equiv \pm m' \bmod n$ for some sign. It is easy to show that if p denotes the number of distinct prime divisors of n, then the number of distinct σ_m is equal to 2^{p-1}. Also note that $\sigma_m \circ \sigma_{m'} = \sigma_{mm'}$ so that all automorphisms commute and are of order 2. All but σ_1 are exceptional. We want to show that the σ_m maps are the complete set of automorphism invariants for $B_{\ell,2}$.

The quantum dimensions of $B_{\ell,2}$ are given in [24]:

$$\mathcal{D}(0) = \mathcal{D}(2\omega^1) = 1, \tag{6.3a}$$
$$\mathcal{D}(\omega^1 + \omega^\ell) = \mathcal{D}(\omega^\ell) = \sqrt{n}, \tag{6.3b}$$
$$\mathcal{D}(\nu^i) = 2, \quad \text{for all } 1 \le i \le \ell. \tag{6.3c}$$

Let σ be any automorphism of $B_{\ell,2}$. The eqs (6.3), together with (3.8a),(3.8c), force $\sigma(\lambda) = \lambda$ for $\lambda \in \{0, 2\omega^1, \omega^1+\omega^\ell, \omega^\ell\}$. Write $\sigma(\nu^1) = \nu^m$; the norm condition (3.8a) then reduces to $n-1 \equiv (n-m)m \bmod 2n$, *i.e.* $m^2 \equiv 1 \bmod n$. Now, σ and σ_m have the same action on ω^1 and ω^ℓ. Thus $\sigma_m \circ \sigma$ leaves them both fixed, and must be the identity by step *(i)* of section 5.2, proving $\sigma = \sigma_m$ everywhere. ■

6.2. The D–Series, Level 2

The height here is $n = 2 + h^\vee = 2\ell$. The $\ell + 7$ weights in $P_+(D_{\ell,2})$ will be denoted by

$$\kappa^1,\, \kappa^2,\, \kappa^3,\, \kappa^4 := 0,\; 2\omega^1,\; 2\omega^\ell,\; 2\omega^{\ell-1}, \tag{6.4a}$$
$$\mu^1,\, \mu^2,\, \mu^3,\, \mu^4 := \omega^{\ell-1},\; \omega^\ell,\; \omega^1 + \omega^{\ell-1},\; \omega^1 + \omega^\ell, \tag{6.4b}$$
$$\nu^i := \omega^i, \quad \text{for } 1 \le i \le \ell - 2, \quad \text{and} \quad \nu^{\ell-1} := \omega^{\ell-1} + \omega^\ell. \tag{6.4c}$$

For each m satisfying $m^2 \equiv 1 \bmod 4\ell$, we define a mapping σ_m on $P_+(D_{\ell,2})$ by

$$\sigma_m \;:\; \begin{cases} \sigma_m(\nu^i) = \nu^{[mi]_n} & \text{for all } 1 \le i \le \ell - 1, \\ \sigma_m(\lambda) = \lambda & \text{if } \lambda \in \{\kappa^i,\, \mu^i\}_{1 \le i \le 4}, \end{cases} \tag{6.5a}$$

with the same definition of $[x]_n$ as in the previous subsection. It will follow from section 6.3 that all σ_m are generalized Galois automorphisms, and as such, that they define automorphism invariants.

Our task in this subsection is to prove the following. For $\ell = 4$, there are precisely six automorphism invariants, namely the six conjugations C_i (all σ_m are trivial in this case). For $\ell > 4$, any automorphism invariant of $D_{\ell,2}$ equals $C_1^a \sigma_m$ for $a = 0, 1$, and σ_m as in (6.5a). Moreover, $C_1^a \sigma_m = C_1^{a'} \sigma_{m'}$ iff both $a \equiv a'$ mod 2 and $m \equiv \pm m'$ mod 2ℓ for some choice of sign.

Thus the number of automorphism invariants for $\ell > 4$ is precisely 2^p, where p is the number of distinct prime divisors of ℓ. When $\ell \not\equiv 2 \bmod 4$, all but two of these, namely σ_1 and C_1, are exceptional ($\sigma_s = \sigma_c = C_1$ and $\sigma_{vsc} = \sigma_1$); when $\ell \equiv 2 \bmod 4$, all but four of them, namely $C_1^a \sigma_{\ell-1}^b$, are ($\sigma_s = \sigma_c = \sigma_{\ell-1}$). Note that for $\ell > 4$, the composition law is

$$C_1^a \sigma_m \circ C_1^{a'} \sigma_{m'} = C_1^{a+a'} \sigma_{mm'}, \tag{6.5b}$$

so that the automorphisms are all of order 2 and commute.

Let us begin with $D_{4,2}$. Computing the norms, we find that ω^1, ω^3, and ω^4 are the only weights in the alcôve with norm equal to 5 mod 16, and ω^2 is the only one with norm equal to 10 mod 16. Therefore any automorphism σ must fix ω^2 and permute ω^1, ω^3, and ω^4. Thus for one of the conjugations C_i of D_4, $C_i \circ \sigma$ fixes all the ω^j, so must equal the identity by Lemma 1.

The quantum dimensions for $D_{\ell,2}$ are computed in [24]:

$$\mathcal{D}(\kappa^i) = 1 \qquad \text{for } 1 \le i \le 4, \tag{6.6a}$$
$$\mathcal{D}(\mu^i) = \sqrt{\ell} \qquad \text{for } 1 \le i \le 4, \tag{6.6b}$$
$$\mathcal{D}(\nu^i) = 2 \qquad \text{for } 1 \le i \le \ell - 1. \tag{6.6c}$$

For $\ell > 4$, $\mathcal{D}(\kappa^i) < \mathcal{D}(\nu^j) < \mathcal{D}(\mu^k)$, so that the three sets of weights must be stable under any σ. Computing the norms, we find that $\sigma\{\mu^1, \mu^2\} = \{\mu^1, \mu^2\}$, so replacing σ by $C_1 \circ \sigma$ if necessary, we may assume $\sigma(\mu^2) = \mu^2$. The mapping $\sigma(\nu^1) = \nu^m$ is allowed by the norm condition (3.8a) only if m satisfies $m^2 \equiv 1 \bmod 4\ell$; since σ and σ_m coincide on $\{\omega^1, \omega^\ell\}$, they coincide everywhere. ■

6.3. Galois and the Level 2 Exceptionals

It is very tempting to interpret the automorphisms σ_m in (6.2) and (6.5a) as pure Galois automorphisms, but in fact not all are. For $B_{\ell,2}$, the Galois group (over $\mathbb{Q}$) of the extension $\mathbb{Q}(S_{\lambda,\lambda'})$ is contained in $\mathbb{Z}^*_{4n}$. Recall that for a Galois transformation g_a to define an automorphism invariant, it has to fix the identity, $g_a(0) = 0$, and to commute with the modular matrix T. Leaving the T–condition aside for the moment, let us look at the other one, and let us suppose that the permutation of the alcôve induced by g_a fixes the identity. If that is so, g_a must leave the quantum dimensions (6.3) invariant, and in particular $g_a(\sqrt{n}) = \sqrt{n}$. It is a standard result in number theory [19] that $g_a(\sqrt{n}) = (n/a)_J \sqrt{n}$, where $(./.)_J$ is the Jacobi symbol, defined in terms of the Legendre symbol and the prime decomposition of a by:

$$\Big(\frac{n}{a}\Big)_J = \prod_p \Big(\frac{n}{p}\Big)_L^{k_p}, \qquad \text{for } a = \textstyle\prod_p p^{k_p}. \tag{6.7}$$

We conclude that g_a can define an automorphism invariant only if a satisfies $(n/a)_J = +1$, and one can show that, together with the T–condition, this is also a sufficient condition. It is now an easy matter to show that the norm condition alone, which roughly speaking amounts to $a^2 \equiv 1 \bmod n$, is not sufficient to guarantee that $(n/a)_J = +1$. If however $(n/a)_J = +1$ is satisfied, then $\sigma_m(\lambda) = g_a(\lambda)$ is a Galois automorphism invariant upon setting $m \equiv a \bmod n$. Similar conclusions apply to $D_{\ell,2}$: the T–condition and $(q/a)_J \cdot (a/2^t)_J = +1$, where $\ell = q \cdot 2^t$ and q odd, are necessary and sufficient conditions for g_a to define a pure Galois automorphism, which then equals σ_m upon setting $m \equiv a \bmod n$.

However one can show that both (6.2) and (6.5a) have the generalized Galois form (3.11). Whenever $S^2_{0,0} \in \mathbb{Q}$ (this is satisfied by $B_{\ell,2}$ and $D_{\ell,2}$ with $S^2_{0,0} = 1/4n$ in both cases), then

$$g(S_{0,0}) = \pm S_{0,0} = \epsilon_g(0)\, S_{g(0),0} \tag{6.8}$$

for any element g of the Galois group, and therefore $g(0) = J(0)$ for some simple current J. For $D_{\ell,2}$, $J = id.$ or J_v, because $\mathcal{D}(\omega^1) \in \mathbb{Q}$ and (3.13) imply $Q_J(\omega^1) \in \mathbb{Z}$. To commute with T, g_a must obey $a^2 \equiv 1 \bmod 2nN$ where $N = 1, 2, 4$ for $\ell \equiv 0, 2, \pm 1 \bmod 4$, respectively. On the other hand, the Galois group for the orthogonal series is $\mathbb{Z}^*_{Mn}$ where $M = 2, 4$ when ℓ is even, odd, respectively. Now for any m obeying $m^2 \equiv 1 \bmod n$ (for $B_{\ell,2}$) or mod $2n$ (for $D_{\ell,2}$), it is easy to verify that an $a \in \mathbb{Z}^*_{Mn}$ can be found such that $a \equiv m \bmod n$ and g_a commutes with T. We want to show $\sigma_m = \sigma_{g_a}$, up to a conjugation.

First note that for $B_{\ell,2}$, any automorphism σ must fix ω^ℓ (see section 6.1), and for $D_{\ell,2}$, either σ or $C_1 \circ \sigma$ will fix ω^ℓ (see section 6.2). Therefore $\sigma_m(\omega^\ell) = C' \circ \sigma_{g_a}(\omega^\ell)$ for some conjugation C'. By step *(i)* of sections 5.2 and 5.4, it suffices to show $\sigma_m(\omega^1) = C' \circ \sigma_{g_a}(\omega^1)$. This can be seen from the following formulas

$$\text{for } B_{\ell,2}\ : \qquad \frac{S_{\omega^1,\nu^j}}{S_{0,\nu^j}} = 2\cos\Big(\frac{2\pi j}{n}\Big), \tag{6.9a}$$

$$\text{for } D_{\ell,2}\ : \qquad \frac{S_{\omega^1,\nu^j}}{S_{0,\nu^j}} = 2\cos\Big(\frac{\pi j}{\ell}\Big). \tag{6.9b}$$

Eq. (6.9a) can be found in [24], while (6.9b) can be derived directly from the formula

$$S_{\omega^1,\lambda} = S_{0,\lambda} \sum_{\mu \in P(\omega^1)} \exp[-2\pi i \mu \cdot (\lambda + \rho)/n], \tag{6.10}$$

where $P(\omega^1)$ is the weight diagram of the defining representation of B_ℓ and D_ℓ. Clearly we have $2 = g_a(\mathcal{D}(\omega^1)) = \mathcal{D}(g_a(\omega^1))$, so that $g_a(\omega^1) = \nu^j$ for some j. Applying g_a to (6.9a) and (6.9b) yields $g_a(\omega^1) = \nu^{[a]_n}$, and consequently $C' \circ \sigma_{g_a}(\omega^1) = C' \circ J(g_a(\omega^1)) = \nu^{[a]_n}$, as claimed.

So we have proved that, up to conjugation, all automorphisms of $B_{\ell,2}$ and $D_{\ell,2}$ are generalized Galois automorphisms of the form (3.11). Let us also mention that in some cases, all of them can in fact be realized as pure Galois automorphisms as well as generalized ones. The reason for this is that, to a single m satisfying $m^2 \equiv 1 \bmod n$ or $2n$, there are in general several $a \in \mathbb{Z}^*_{Mn}$ such that $a \equiv m \bmod n$ and g_a commutes with T. This happens for instance when ℓ is odd. For $B_{\ell,2}$, ℓ odd, g_a commutes with T if $a^2 \equiv 1 \bmod 8n$. But then $a' = a + 2n$ also satisfies $a'^2 \equiv 1 \bmod 8n$, and both a and a' lead to the same m by reduction modulo n. They however make a difference because $(n/a')_J = (-1)^{\ell n}(n/a)_J$, so that, for ℓ odd, either g_a or $g_{a'}$ fixes the identity. Hence $\sigma_m = \sigma_{g_a}$ or $\sigma_{g_{a'}}$ is a pure Galois automorphism. The same conclusion holds for $D_{\ell,2}$ when $\ell \equiv 3 \bmod 4$.

7. The Exceptional Algebras

We complete in this section the proof of the Theorem for the five exceptional simple Lie algebras. Fusion rules will be most easily presented by writing decompositions of tensor products of finite Lie algebra representations, since fusion coefficients are identical to the coefficients of truncated tensor products [8,25], with the truncation related in a simple way to the level. Explicitly, we can write

$$\lambda \otimes \lambda' = \bigoplus_{\mu} \bigoplus_{k_t} \; m(k_t)^{\mu}_{\lambda,\lambda'} \; \left(\mu\right)_{k_t}, \tag{7.1a}$$

with the fusion coefficient $N^{\mu}_{\lambda,\lambda'}$ at level k determined by

$$N^{\mu}_{\lambda,\lambda'} \;=\; \sum_{k_t=0}^{k} \; m(k_t)^{\mu}_{\lambda,\lambda'} \quad . \tag{7.1b}$$

(7.1a) presents the fusion rules for all levels in an economical way. It is just the tensor product $\lambda \otimes \lambda'$, with the representations in the decomposition labelled by the threshold level k_t at and above which the corresponding affine representation appears in the corresponding fusion rule. In particular, the tensor product coefficients (or Clebsch-Gordan series coefficients) are

$$R^{\mu}_{\lambda,\lambda'} \;=\; \sum_{k_t=0}^{\infty} m(k_t)^{\mu}_{\lambda,\lambda'} \quad . \tag{7.1c}$$

For convenience we will also include a superscript indicating the "norm squared" of the highest weight of the representations in the tensor product decomposition:

$$\lambda \otimes \lambda' = \bigoplus_{\mu} \bigoplus_{k_t} \; m(k_t)^{\mu}_{\lambda,\lambda'} \; \left(\mu\right)^{n(\mu)}_{k_t}, \qquad \text{with } n(\mu) := (\rho+\mu)^2. \tag{7.1d}$$

7.1. The Algebra E_6

The colabels of E_6 are equal to $(1,2,3,2,1,2)$, so that a weight of $P_+(E_{6,k})$ satisfies $\lambda_0 + \lambda_1 + 2\lambda_2 + 3\lambda_3 + 2\lambda_4 + \lambda_5 + 2\lambda_6 = k$ and the dual Coxeter number is equal to 12. The charge conjugation acts as $C(\lambda) = (\lambda_0; \lambda_5, \lambda_4, \lambda_3, \lambda_2, \lambda_1, \lambda_6)$.

There is a simple current J of order 3, given by $J\lambda = (\lambda_5; \lambda_0, \lambda_6, \lambda_3, \lambda_2, \lambda_1, \lambda_4)$. Its charge and weight are $Q_J(\lambda) = (-\lambda_1 + \lambda_2 - \lambda_4 + \lambda_5)/3$ and $h_{J(0)} = 2k/3$. When k is coprime with 3, there is a simple current automorphism invariant [1]

$$\sigma_J(\lambda) = J^{k(\lambda_1-\lambda_2+\lambda_4-\lambda_5)}(\lambda), \qquad \text{if } (k,3) = 1. \tag{7.2}$$

Note that for $k = 1, 2$, $\sigma_J = C$.

When $k \equiv 0 \bmod 3$, there are only two automorphism invariants, σ_1 and C, and there are two more when $k \geq 4$ and 3 are coprime, namely σ_J and $C\sigma_J$ (σ_J is of order 2, $\sigma_J^2 = \sigma_1$).

Let σ be any automorphism invariant of $E_{6,k}$. From the corollary of section 4, we have that, for any k, $\sigma(\omega^1) = C^a J^b(\omega^1)$ for some $a = 0, 1$, $b = 0, 1, 2$. Replacing σ with $C^a \circ \sigma$, we may assume $a = 0$.

Consider first $b = 1$. The norm condition yields

$$(\rho + \omega^1)^2 \equiv (\rho + J\omega^1)^2 \mod 2n \quad \Rightarrow \quad \frac{4}{3}\,(k-1)\,n \equiv 0 \mod 2n. \tag{7.3}$$

Therefore $\sigma(\omega^1) = J(\omega^1)$ requires $k \equiv 1 \bmod 3$. But for precisely these k, the automorphism invariant σ_J has the same action on ω^1, $\sigma_J(\omega^1) = J(\omega^1)$. Replacing σ with $\sigma_J \circ \sigma$ for these k, we may assume $b = 0$. The argument for $b = 2$ is identical, so that for all k, we may assume $\sigma(\omega^1) = \omega^1$, and prove that the only automorphism with that property is σ_1.

Consider first the finite-dimensional Lie algebra tensor product

$$\omega^1 \otimes \omega^1 = \big(\omega^2\big)_2^{111\frac{1}{3}} \ \oplus\ \big(\omega^5\big)_1^{95\frac{1}{3}} \ \oplus\ \big(2\omega^1\big)_2^{115\frac{1}{3}}, \tag{7.4a}$$

with subscript threshold levels indicating the corresponding fusions. Since ω^1 is fixed by σ, the weights on the r.h.s. of (7.4a) must be permuted. However, from the superscripts, we read that the norms are all different, so that the permutation must be trivial. In particular ω^2 and ω^5 must be fixed as well. By considering in a similar way the following sequence of tensor products, we can establish that all fundamental weights of E_6 are fixed by σ, for all levels $k \geq 1$. The following three tensor product decompositions are sufficient:

$$\omega^1 \otimes \omega^5 = \big(0\big)_1^{78} \ \oplus\ \big(\omega^6\big)_2^{102} \ \oplus\ \big(\omega^1 + \omega^5\big)_2^{114}\ , \tag{7.4b}$$

$$\omega^1 \otimes \omega^2 = \big(\omega^3\big)_3^{126} \ \oplus\ \big(\omega^6\big)_2^{102} \ \oplus\ \big(\omega^1 + \omega^2\big)_3^{132} \ \oplus\ \big(\omega^1 + \omega^5\big)_2^{114}\ , \tag{7.4c}$$

$$\omega^1 \otimes \omega^6 = \big(\omega^1\big)_2^{95\frac{1}{3}} \ \oplus\ \big(\omega^4\big)_2^{111\frac{1}{3}} \ \oplus\ \big(\omega^1 + \omega^6\big)_3^{121\frac{1}{3}}\ . \tag{7.4d}$$

We note that ω^i, $i \neq 1$, appears in the fusions (7.4) if and only if the value of k allows it to be in $P_+(E_{6,k})$. Thus all fundamental weights in the alcôve must be fixed by σ, and by Lemma 1, this shows that $\sigma = \sigma_1$ as soon as $\sigma(\omega^1) = \omega^1$, and the proof is complete. ∎

7.2. The Algebra E_7

A weight in the alcôve satisfies $\lambda_0 + 2\lambda_1 + 3\lambda_2 + 4\lambda_3 + 3\lambda_4 + 2\lambda_5 + \lambda_6 + 2\lambda_7 = k$, and the dual Coxeter number is $h^\vee = 18$. The charge conjugation is trivial, but there is a simple current J of order 2, given by $J(\lambda) = (\lambda_6; \lambda_5, \ldots, \lambda_1, \lambda_0, \lambda_7)$. It has $Q_J(\lambda) = (\lambda_4 + \lambda_6 + \lambda_7)/2$ and $h_J = 3k/4$. When $k \equiv 2 \bmod 4$, it gives rise to the simple current automorphism invariant [3]

$$\sigma_J(\lambda) = J^{\lambda_4 + \lambda_6 + \lambda_7}(\lambda), \qquad \text{if } k \equiv 2 \bmod 4. \tag{7.5}$$

We note that $\sigma_J = \sigma_1$ for $k = 2$.

There is only the trivial automorphism $\sigma = \sigma_1$ when $k \not\equiv 2 \bmod 4$ or $k = 2$, and for $k > 2$ and $k \equiv 2 \bmod 4$, there are two, σ_1 and σ_J.

From the corollary of section 4, we know that, for any automorphism and any value of k, $\sigma(\omega^6) = J^b(\omega^6)$ for some $b = 0, 1$. Suppose $b = 1$. Then

$$(\rho + \omega^6)^2 \equiv (\rho + J\omega^6)^2 \mod 2n \quad \Rightarrow \quad (\frac{3}{2}k - 3)\,n \equiv 0 \mod 2n. \tag{7.6}$$

Thus $\sigma(\omega^6) = J(\omega^6)$ requires $k \equiv 2 \bmod 4$. Precisely for these k, the automorphism invariant σ_J has the property that $\sigma_J(\omega^6) = J(\omega^6)$. Replacing σ with $\sigma_J \circ \sigma$, we may assume $b = 0$. Thus for all k, we may assume $\sigma(\omega^6) = \omega^6$, and prove that the only such automorphism is σ_1.

For $k \geq 1$, all fundamental weights ω^i belonging to $P_+(E_{7,k})$ appear in the fusions (7.7) and the usual argument shows that they must be fixed by σ. The fusion threshold levels of the following tensor products [33] were obtained using the affine Weyl group:

$$\omega^6 \otimes \omega^6 = \big(0\big)_1^{199.5} \oplus \big(\omega^1\big)_2^{235.5} \oplus \big(\omega^5\big)_2^{255.5} \oplus \big(2\omega^6\big)_2^{259.5}, \tag{7.7a}$$

$$\omega^1 \otimes \omega^6 = \big(\omega^6\big)_2^{228} \oplus \big(\omega^7\big)_2^{252} \oplus \big(\omega^1 + \omega^6\big)_3^{266}, \tag{7.7b}$$

$$\omega^1 \otimes \omega^1 = \big(0\big)_2^{199.5} \oplus \big(\omega^1\big)_3^{235.5} \oplus \big(\omega^2\big)_3^{271.5} \oplus \big(\omega^5\big)_2^{255.5} \oplus \big(2\omega^1\big)_4^{275.5}, \tag{7.7c}$$

$$\omega^1 \otimes \omega^7 = \big(\omega^4\big)_3^{282} \oplus \big(\omega^6\big)_2^{228} \oplus \big(\omega^7\big)_3^{252} \oplus \big(\omega^1 + \omega^6\big)_3^{266} \oplus \big(\omega^1 + \omega^7\big)_4^{292}, \tag{7.7d}$$

$$\begin{aligned} \omega^7 \otimes \omega^7 = {} & \big(0\big)_2^{199.5} \oplus \big(\omega^1\big)_3^{235.5} \oplus \big(\omega^2\big)_3^{271.5} \oplus \big(\omega^3\big)_4^{307.5} \oplus \big(\omega^5\big)_3^{255.5} \oplus \big(2\omega^1\big)_4^{275.5} \\ & \oplus \big(2\omega^6\big)_2^{259.5} \oplus \big(2\omega^7\big)_4^{311.5} \oplus \big(\omega^1 + \omega^5\big)_4^{295.5} \oplus \big(\omega^6 + \omega^7\big)_3^{283.5}. \end{aligned} \tag{7.7e}$$

Going through these five products and assuming $\sigma(\omega^6) = \omega^6$, we have successively that ω^1, ω^5, ω^7, ω^2, ω^4 and ω^3 must be fixed by σ. By Lemma 1, the whole of the alcôve must be fixed. ■

7.3. The Algebra E_8

Here a weight satisfies $\lambda_0 + 2\lambda_1 + 3\lambda_2 + 4\lambda_3 + 5\lambda_4 + 6\lambda_5 + 4\lambda_6 + 2\lambda_7 + 3\lambda_8 = k$, and the dual Coxeter number is 30. The charge conjugation is trivial and there is no simple current (except for an anomalous one at $k = 2$ which does not give rise to an automorphism invariant).

We will show that for all levels $k \neq 4$, there is only the trivial automorphism invariant σ_1, and that for $k = 4$, there is a second, exceptional one we call σ_{e8}, and which was first given in [10]. It permutes the fundamental weights ω^1 and ω^6 and fixes all other weights:

$$\sigma_{e8} \;:\; \begin{cases} \omega^1 \longleftrightarrow \omega^6, \\ \text{fixes all other weights.} \end{cases} \tag{7.8}$$

This is *not* a Galois automorphism: the Galois group for $E_{8,k}$ is $\mathbb{Z}^*_{k+30}$; g_a commutes with T iff $a^2 \equiv 1 \bmod n$; so the only possible g_a at $k = 4$ are g_1 and g_{-1}, both of which give σ_1. Remarkably, in the set of all automorphism invariants for all simple X_ℓ and all levels k, σ_{e8} is the *only one* that cannot be explained in terms of simple currents, conjugations, Galois transformations or these combined.

Let σ be any automorphism invariant of $E_{8,k}$. From the corollary of section 4, we have that, for any $k \neq 4$, $\sigma(\omega^1) = \omega^1$. For $k = 4$, the only other weight in the alcôve which has the same quantum dimension as ω^1 is ω^6, so that the additional possibility is $\sigma(\omega^1) = \omega^6$. But in this case we can replace σ with $\sigma_{e8} \circ \sigma$ so that the new σ fixes ω^1. Thus for all k, we may assume $\sigma(\omega^1) = \omega^1$. The proof will be complete if we show that any such automorphism is necessarily trivial.

We show that if ω^1 is fixed, then so are all ω^i, for $1 \leq i \leq 8$, which are in the alcôve. By the usual norm argument, the fusions encoded in the following sequence of tensor products establishes the result *except* for ω^5:

$$\omega^1 \otimes \omega^1 = \big(0\big)_2^{620} \oplus \big(\omega^1\big)_3^{680} \oplus \big(\omega^2\big)_3^{740} \oplus \big(\omega^7\big)_2^{716} \oplus \big(2\omega^1\big)_4^{744}, \tag{7.9a}$$

$$\omega^1 \otimes \omega^7 = \big(\omega^1\big)_2^{680} \oplus \big(\omega^2\big)_3^{740} \oplus \big(\omega^7\big)_3^{716} \oplus \big(\omega^8\big)_3^{764} \oplus \big(\omega^1 + \omega^7\big)_4^{780}, \tag{7.9b}$$

$$\omega^1 \otimes \omega^8 = \big(\omega^2\big)_3^{740} \oplus \big(\omega^3\big)_4^{800} \oplus \big(\omega^6\big)_4^{816} \oplus \big(\omega^7\big)_3^{716} \oplus \big(\omega^8\big)_4^{764}$$

$$\oplus\ \big(\omega^1+\omega^7\big)_4^{780}\ \oplus\ \big(\omega^1+\omega^8\big)_5^{830}\ , \tag{7.9c}$$

$$\omega^1\otimes\omega^3 = \big(\omega^2\big)_4^{740}\ \oplus\ \big(\omega^3\big)_5^{800}\ \oplus\ \big(\omega^4\big)_5^{860}\ \oplus\ \big(\omega^6\big)_4^{816}\ \oplus\ \big(\omega^8\big)_4^{764}\ \oplus\ \big(\omega^1+\omega^2\big)_5^{806}$$

$$\oplus\ \big(\omega^1+\omega^3\big)_6^{868}\ \oplus\ \big(\omega^1+\omega^7\big)_4^{780}\ \oplus\ \big(\omega^1+\omega^8\big)_5^{830}\ \oplus\ \big(\omega^2+\omega^7\big)_5^{844}\ . \tag{7.9d}$$

These fusions were calculated from the corresponding tensor products listed in [26] using the affine Weyl group.

The remaining fundamental representation, ω^5, is contained in the following tensor product:

$$\omega^6\otimes\omega^7 = \big(\omega^2\big)_4^{740}\ \oplus\ \big(\omega^3\big)_5^{800}\ \oplus\ \big(\omega^4\big)_5^{860}\ \oplus\ \big(\omega^5\big)_6^{920}\ \oplus\ 2\big(\omega^6\big)_{4,5}^{816}\ \oplus\ \big(\omega^7\big)_4^{716}$$

$$\oplus\ \big(\omega^8\big)_4^{764}\ \oplus\ \big(\omega^1+\omega^2\big)_5^{806}\ \oplus\ \big(\omega^1+\omega^3\big)_6^{868}\ \oplus\ \big(\omega^1+\omega^6\big)_6^{884}\ \oplus\ 2\big(\omega^1+\omega^7\big)_{4,5}^{780}$$

$$\oplus\ 2\big(\omega^1+\omega^8\big)_{5,5}^{830}\ \oplus\ 2\big(\omega^2+\omega^7\big)_{5,5}^{844}\ \oplus\ \big(\omega^2+\omega^8\big)_6^{896}\ \oplus\ \big(\omega^3+\omega^7\big)_6^{908}\ \oplus\ \big(\omega^6+\omega^7\big)_6^{926}$$

$$\oplus\ 2\big(\omega^7+\omega^8\big)_{5,5}^{870}\ \oplus\ \big(2\omega^7\big)_5^{820}\ \oplus\ \big(\omega^1+2\omega^7\big)_6^{888}\ \oplus\ \big(2\omega^1+\omega^7\big)_6^{848}\ . \tag{7.9e}$$

This time, the norm argument is not sufficient to show from (7.9e) that σ must also fix ω^5. However, one can show that the only representations that can possibly be exchanged with ω^5 are $\big(2\omega^7\big)_5^{820}$ and $\big(2\omega^1+\omega^7\big)_6^{848}$, and that can only happen for levels $k=20$ and $k=6$, respectively. But it is then easily checked for these levels that their quantum dimensions differ, so that at the end ω^5 too must be fixed. Thus, for all $k\geq 2$, all fundamentals ω^i in the alcôve must be fixed if ω^1 is fixed, and Lemma 1 implies once more that $\sigma=\sigma_1$, completing the proof for E_8. ■

7.4. The Algebra F_4

A weight in $P_+(F_{4,k})$ satisfies $\lambda_0+2\lambda_1+3\lambda_2+2\lambda_3+\lambda_4=k$, and the dual Coxeter number is $h^\vee=9$. The charge conjugation C is trivial, and there is no simple current.

We will show that for $k\neq 3$, σ_1 is the only automorphism invariant, and that at $k=3$, there is one more, namely the exceptional σ_{f4}, first found in [35]. It is given by

$$\sigma_{f4}\ :\ \begin{cases}\text{permutes } \omega^2\longleftrightarrow\omega^4 \quad\text{and}\quad \omega^1\longleftrightarrow 3\omega^4,\\ \text{fixes all other weights.}\end{cases} \tag{7.10}$$

In fact this permutation equals the one induced by the Galois transformation g_5, given by (3.11) (a pure Galois automorphism). For $k=3$, the relevant Galois group is isomorphic to $\mathbb{Z}_{24}^*$, and one finds, in the notation of section 3, that $g_5(\lambda)=\sigma_{f4}(\lambda)$.

From the corollary of section 4, we have that for $k\neq 3$, an automorphism must satisfy $\sigma(\omega^4)=\omega^4$. From Table 3, the only other possibility at $k=3$ is $\sigma(\omega^4)=\omega^2$, but in this case, we can replace σ by $\sigma_{f4}\circ\sigma$ and assume that ω^4 is fixed. The conclusion follows if we show that $\sigma(\omega^4)=\omega^4$ implies $\sigma=\sigma_1$.

This is easily done with the following two tensor products

$$\omega^4\otimes\omega^4 = \big(0\big)_1^{39}\ \oplus\ \big(\omega^1\big)_2^{57}\ \oplus\ \big(\omega^3\big)_2^{63}\ \oplus\ \big(\omega^4\big)_1^{51}\ \oplus\ \big(2\omega^4\big)_2^{65}\ , \tag{7.11a}$$

$$\omega^3\otimes\omega^4 = \big(\omega^1\big)_2^{57}\ \oplus\ \big(\omega^2\big)_3^{75}\ \oplus\ \big(\omega^3\big)_2^{63}\ \oplus\ \big(\omega^4\big)_2^{51}$$

$$\oplus\ \big(\omega^1+\omega^4\big)_3^{71}\ \oplus\ \big(\omega^3+\omega^4\big)_3^{78}\ \oplus\ \big(2\omega^4\big)_2^{65}\ . \tag{7.11b}$$

The norm condition implies that all representations in these two products must be fixed by σ if ω^4 is fixed, so in particular those fundamentals lying in the alcôve are fixed, implying $\sigma=\sigma_1$ by Lemma 1. ■

7.5. The Algebra G_2

A weight in the alcôve satisfies $\lambda_0 + 2\lambda_1 + \lambda_2 = k$, and $h^\vee = 4$. There is no charge conjugation nor simple current.

The only non–trivial automorphism invariant σ_{g2} is found at level $k = 4$ [35]. It is the following permutation

$$\sigma_{g2} \ : \ \begin{cases} \text{permutes } \omega^1 \longleftrightarrow 4\omega^2 \quad \text{and} \quad 2\omega^1 \longleftrightarrow \omega^2, \\ \text{fixes all other weights.} \end{cases} \tag{7.12}$$

The Galois group $\mathbb{Z}_{24}^*$ is the same as for $F_{4,3}$, and $g_5(\lambda) = \sigma_{g2}(\lambda)$ also holds here.

From the corollary of section 4, the second fundamental weight ω^2 must be left invariant by any σ, for $k \neq 4$. At $k = 4$, Table 3 shows that the only other possibility is $\sigma(\omega^2) = 2\omega^1$; in this case, composing σ with σ_{g2} allows to assume that, here too, ω^2 must be fixed.

It remains to show that an automorphism fixing ω^2 is trivial. This immediately follows from the tensor product

$$\omega^2 \otimes \omega^2 = \big(0\big)_1^{4\frac{2}{3}} \ \oplus \ \big(\omega^1\big)_2^{12\frac{2}{3}} \ \oplus \ \big(\omega^2\big)_1^{8\frac{2}{3}} \ \oplus \ \big(2\omega^2\big)_2^{14}\,, \tag{7.13}$$

which shows that ω^1 is fixed, and from Lemma 1. ■

8. Conclusion

In this paper, we have established the complete list of automorphism modular invariants for unextended current models based on finite–dimensional simple Lie algebras. More precisely, we have classified the modular invariant forms, sesquilinear in the affine characters, obtained by twisting the diagonal invariant by an automorphism of the fusion ring. Some of these invariants correspond to torus partition functions of WZW models [13]. In particular, the diagonal invariants describe WZW models based on simply-connected simple Lie groups. The WZW partition functions for nonsimply-connected groups can be obtained by "orbifolding" the diagonal invariant [18]. Many of our invariants can be obtained by similarly "orbifolding" the conjugation invariant. But many others await interpretation. For example, those Galois automorphism invariants which cannot be written as simple current invariants seem problematic at present. Another interesting case is provided by the exceptional invariant of $E_{8,4}$ which, in the list of invariants for all simple algebras and all levels, is the only one that cannot be described in terms of simple currents, conjugations and/or Galois transformations.

Although the list of automorphism invariants constitutes major progress towards the general problem of classifying modular invariants for conformal current models, more technical problems need to be overcome before the full list of modular invariants can be contemplated. Among the most striking ones, let us mention the fixed point resolution problem, and more importantly, the classification of the chiral extensions of Kac–Moody algebras. A humbler task should be to extend our analysis to the remaining semi-simple Lie algebras — something of direct value for the Goddard–Kent–Olive models.

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